Part Four

Arithmetic

In reading, we teach the child the meaning and uses of the twenty-six alphabet symbols of which all our written language is composed. In writing, we teach him to write these symbols. In arithmetic we teach the child the meaning and uses of the ten numeral symbols—a set of symbols as important to the development of our civilization as the alphabet. All arithmetic calculation is performed with these ten remarkable symbols.

Like the alphabet, our decimal place-value arithmetic system is the product of a very long evolution in which man sought to find the best mental tools with which to express the tremendous intellectual potential he had within him. What man could do, provided he had the right tools and social conditions, can be measured by the incredible leap he made from the primitive agricultural civilization of the 1500s to the space-age civilization of the 1970s—a span of less than 500 years. Of that leap forward, the greatest progress was made in the last two hundred years. This is an unbelievably short time when measured against the total span of man's existence on earth. It indicates that for a long time man had the intellectual potential to make that leap, but for centuries he lacked the mental tools with which to realize it. The two necessary tools were the alphabet and the decimal place-value arithmetic system. Western man developed an alphabet about 1000 B.C. But he had to wait another two thousand years before he got the most effective mental tool for arithmetic calculation ever devised by the human race. Again, its hallmark, its genius, was simplicity—a simplicity which took centuries to develop.

In about 1000 B.C. came the first tool to provide the great intellectual leap forward—the alphabet. The Greeks adopted it from the Phoenicians and developed their highly literate civilization which marks the beginning of Western ascendancy. While the alphabet promoted the literary arts, philosophy, science, and mathematics, the Greek method of arithmetic was not much better than the Egyptian. It used no less than twenty-seven symbols—the twenty-four letters of the Greek alphabet plus three archaic symbols—and made no use of place value. It was a purely additive numeral notation. The first nine letters represented 1 through 9; the next nine represented the tens from 10 through 90; and the last nine represented the hundreds from 100 through 900. Thus the number 125 required a symbol for the hundred, a symbol for the twenty, and a symbol for the five. The symbols, regardless of their order, all added up to 125. Note that our number 125 adds up to 8. We understand its meaning because of place value or positional value.

While the Greek system was easy for addition and subtraction, it was difficult for multiplication, division, and fractions. To facilitate calculation, the Greeks, as well as everyone else in the ancient world, required the help of an abacus, a counting device derived from the primitive system of counting pebbles. In fact, the word calculate is derived from *calx*, the Latin word for pebble. Thus, in ancient Greece and in Europe right up to the 14th century, specialists had to be trained to do the kind of arithmetic calculation which today is performed by school children with little difficulty. Yet, the Greeks were able to develop Euclidean geometry and make other significant advances in mathematics. How was this possible? Here we must make the distinction between arithmetic, that is, the art of counting and calculation, and mathematics, the science or philosophy of relationships. In Greek times arithmetic was called *logistic* and was considered outside the domain of science and philosophy. Logistic was the tool of commerce. Mathematics, on the other hand, dealt with the abstractions of numbers and the relationships of geometric forms, most of which could be discussed rhetorically, without the use of arithmetic calculation. Euclid used no arithmetic or logistic at all and exercised a strict taboo against using it, while other mathematicians like Archimedes and Heron used it at will without philosophical prejudice. Mathematics was considered the purest form of philosophy, to be studied apart from any consideration for its practical use. Thus, while Greece made great advances in philosophical thought, its arithmetic was just as primitive as that of any other nation. On this subject, Tobias Dantzig*observed

> One who reflects upon the history of reckoning up to the invention of the principle of position is struck by the paucity of achievement. This long period of nearly five thousand years saw the fall and rise of many a civilization, each leaving behind it a heritage of literature, art, philosophy and religion. But what was the net achievement in the field of reckoning, the earliest art practiced by man? An inflexible numeration so crude as to make progress well-nigh impossible, and a calculating device so limited in scope that even elementary calculations called for the services of an expert. And what is more, man used these devices for thousands of years without making a single worthwhile improvement in the instrument, without contributing a single important idea to the system! . . .
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> When viewed in this light, the achievement of the unknown Hindu who some time in the first centuries of our era discovered the principle of position assumes the proportions of a world event. Not only did this principle constitute a radical departure in method, but we know now that without it no progress in arithmetic was possible. And yet the principle is so simple that today the dullést school boy has no difficulty in grasping it.

The date of the first known appearance of a decimal place-value notation in India is 595 A.D. The oldest reference to the place-value system outside India is found in a work written by Severus

*_Number, The Language of Science_ (New York: Macmillan, 1946).

Sebokht, a Syrian bishop, in 662. His comments about it are interesting:

> I will omit all discussion of the science of the Hindus, a people not the same as the Syrians; their subtle discoveries in this science of astronomy, discoveries that are more ingenious than those of the Greeks and the Babylonians; their valuable methods of calculation; and their computing that surpasses description. I wish only to say that this computation is done by means of nine signs. If those who believe, because they speak Greek, that they have reached the limits of science should know these things they would be convinced that there are also others who know something.

The bishop referred to "nine signs" because the sign for zero had not yet been invented. The Hindus had derived the place value concept from their operation of the counting board in which each column stood consecutively for ones, tens, hundreds, thousands, etc. Somewhere along the line it became necessary to designate a symbol for an empty column so that the reader could tell the difference between 32, 302, 3002. "And so," writes Professor Dantzig, "from all appearances, the discovery of zero was an accident brought about by an attempt to make an unambiguous permanent record of a counting board operation."

We shall never know for certain whether the invention of zero was an accident or a stroke of genius, but it was, according to Dantzig,

> . . . the turning-point in a development without which the progress of modern science, industry, or commerce is inconceivable. And the influence of this great discovery was by no means confined to arithmetic. By paving the way to a generalized number concept, it played as fundamental a role in practically every branch of mathematics. In the history of culture the discovery of zero will always stand out as one of the greatest single achievements of the human race.

How did the Hindu place-value system finally make its way into Western culture? It was carried there by the Arabs who had adopted the Hindu system for their own use and brought it to Europe in their invasion of Spain. It should be noted that all this took place after the rise of Christianity, the disintegration of the Roman Empire, and the subsequent sweep of Islam from Arabia to Gibraltar. The Moors established their rule in Spain in 747, but it took another 500 years before the place-value system finally swept the old Roman numerals and the abacus from Christendom.

Why did it take so long? In the wake of Rome's disintegration, Europe had entered the period known as the Dark Ages, during which all learning and intellectual progress seemed to come to a complete halt. The awakening finally came in the thirteenth century. The first book to systematically introduce the Arab-Hindu system in Europe was Leonardo of Pisa's *Liber Abaci*, published in 1202. But it wasn't until about 1500 that only Arabic numerals were used in commercial account books in Europe and the Roman numerals were discarded altogether. There were a number of reasons why the adoption of the Hindu-Arabic system took so long. The main reason, however, was the difficulty people had in adopting a new set of symbols for their numbers and *in using the symbols themselves in the calculating process.* It was quite difficult for many accountants to make the leap from abacus counting (that is, one-to-one concrete counting) to symbolic or abstract counting (that is, counting by the use of the abstract symbols alone). Symbolic counting won out, however, because it was so much faster and the place-value system permitted all calculation to be done with only ten symbols. It took time before people realized that the key to the most efficient use of this system was memory. Once you memorized the addition, subtraction, multiplication and division tables for the first ten numbers, you could perform any arithmetic calculation on paper with great speed. Once the key role of memory in this system was understood, the problem became one of developing the most efficient techniques of memorization—from which the notion of drills originated.

By the end of the fifteenth century, all the techniques (or algorithms, as mathematicians call them) which we now use to add, subtract, multiply and divide were fully developed. In other words, modern arithmetic is scarcely five hundred years old. Thus the gap between the abacus and the computer is less than five hundred years. It was the New Arithmetic which made our technology possible.

Pierre Laplace, the 18th-century French mathematician, eloquently summed up the importance of the place-value system when he wrote:

> It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to all computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of this achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity.

And thus it is probable that the space age would have never come into being had the place-value system and the symbol for zero not been invented.

In teaching the child arithmetic, it is important to convey to him the genius of the system itself. Next to the alphabet, it is the greatest mental tool ever devised by man. Therefore we should approach the subject with the excitement and interest it deserves. Any teacher who makes arithmetic dull does so because he or she does not understand its beautiful simplicity, its logic, and its facility which permit us to do so much with so little. The ten-symbol place-value system is perhaps the diamond of human intellect. It, and the alphabet are the child's greatest intellectual inheritance. Both sets of symbols represent the distilled genius of the human race. It is therefore obvious that such gifts must be presented in such a way as to make the child appreciate what these symbols can do for him in furthering his own potential and happiness.

At this point, it would be appropriate to discuss the matter of arithmetic and the New Math. Among educators today there is a tremendous confusion between what we mean by arithmetic and what we mean by mathematics. Arithmetic is no longer taught as arithmetic. It has been submerged, fragmentized, and scrambled in a much larger area of study called Elementary Mathematics—more popularly known as the New Math. Because of this, students scarcely become aware of the decimal place-value system as *a complete arithmetic system quite separate and distinct from the rest of the subject matter in elementary mathematics.* The result is that students learn arithmetic very poorly and very haphazardly.

With this tutoring book, the tutor can teach the basic arithmetic system without the distractions of the irrelevant mathematical theories and concepts taught in the New Math. Once the child has mastered the arithmetic system, he'll be in a much better position to deal with the often confusing theories and concepts of the New Math.

There has been some discussion among parents and teachers over the practical value of the New Math to the child. Most of us use a great deal of arithmetic throughout our lives—filling out income tax returns, balancing our checking accounts, buying on credit, figuring out mortgage payments, everyday purchasing at the supermarket, etc. But few of us ever use the algebra, geometry and trigonometry we were taught in school. This is not an argument against teaching algebra, geometry and trigonometry. But it is an argument *for* teaching arithmetic as thoroughly and systematically as possible in the primary grades.

Arithmetic, in the form of our decimal place-value system, is one of the most useful tools a child can learn to master. Since a knowledge of arithmetic is vital to an individual's economic survival and success, it should be given top priority in the curriculum of the elementary school. Unfortunately, arithmetic is considered a sort of stepchild in the house of mathematics and is given little attention by the curriculum developers. If it hadn't been for the ingenious invention of the Hindu place-value system, mathematicians wouldn't even bother to teach arithmetic at all. They would have left it to the counting specialist. As we mentioned earlier, the Greek mathematicians looked down on arithmetic as being unworthy of their attention. In today's primary school curriculum, arithmetic has been mathematized out of existence. Teachers no longer talk about arithmetic. They talk only of mathematics—the word arithmetic having been virtually stricken from the schoolroom vocabulary.

For most people, however, arithmetic, like the alphabet, is considered so simple a concept that we forget how complex it really is. Those of us who went to school in previous generations were taught arithmetic mainly by rote. We learned how to add, subtract, multiply, and divide without any trouble because we were taught to memorize our tables and drilled in them constantly. We knew that the multiplication and long-division methods worked, but we didn't know why they worked or who had invented these methods, or when they had come into use. For all we knew, man had been doing long division since the beginning of time. If something worked, we gave little thought as to why it worked or who first worked it out. Nor did it seem necessary for us to know why it worked or who invented it. The value of the methods we were using was so obvious that other considerations were of no importance. If a child discovers that a dollar bill is worth so much candy in a store, he is not interested in the history of money at that point or the theory of supply and demand. He is much more interested in learning how to get more dollars so that he can exchange them for more candy. If we stopped him in his tracks and told him that he ought to know the history of money before using it, he might find our history quite irrelevant to his immediate pursuit.

The same situation applies to the learning of arithmetic. When the child is being taught the rudiments of addition and subtraction, he is much more interested in the fact that the methods being taught work and permit him to master counting than in why the methods work. The *why* is quite irrelevant at this point in learning. That is why it was possible for children to become so proficient in arithmetic by rote learning. They saw, in the doing, that it *worked*—and they did it.

Today, all that rote learning has been thrown out of the schoolroom window and an attempt is being made to explain to the child how addition, subtraction, multiplication, and division work. Unfortunately, the explanations are not very good and often much too confusing and tiresome, with the result that children neither learn to perform arithmetic well nor understand it.

Therefore, in this book we shall teach the child both to perform his arithmetic well and to understand it, but with the strict proviso that in teaching a child to use a complex abstract system it is not always desirable to precede instruction with understanding. Sometimes such premature "understanding" can, in fact, retard performance. For example, if we taught a child the grammatic structure of our language before he began to speak, he might never speak for fear of being wrong. The child learns to speak completely on his own before he knows anything about grammar, correct pronunciation, parts of speech, or the origin of words. He speaks because he finds out by experience that speaking *works*. He finds himself understood, and he goes on to increase his speaking vocabulary and to improve his pronunciation so that he can make himself better understood. To explain to him at the earliest stages of his mastery *why* language works or *how* it works would be of little help, and in fact, might hinder and confuse him. So we leave the child alone and let him learn to speak on his own, making minor corrections here and there so that he can improve his pronunciation and be better understood.

Sometimes an understanding of a method comes automatically with the learning. Sometimes it has to be given quite separately and for reasons other than learning how to use the method. When we study the history of language or philology, we do so only peripherally to improve our working use of the language. Our main interest is in discovering how the human race developed linguistically. The same is true of arithmetic. When we study the history of arithmetic, it is not to help us use arithmetic better. That can only be done by drilling the tables until we know them cold. We study the history of arithmetic to give us insights into the development of civilization, the development of methods, etc., but only peripherally to improve our working mastery of arithmetic.

Therefore, in teaching the child arithmetic, we should not allow ourselves to confuse the idea of a *working knowledge* with that of *intellectual understanding.* In teaching an abstract system to a child, a working knowledge must precede intellectual understanding. It is on this principle that the child learns to speak. It is on this principle that civilization has advanced. Man progressed from the cave to complex civilization primarily because he imitated what *worked* for his predecessors. Children learn by imitating adults. When they become adults themselves and discover why something works, they make improvements. Their children imitate these improvements and thus a higher level of civilization is reached.

The interesting thing about arithmetic is that until the development of the Hindu decimal place-value system, very little improvement had been made in the art of calculation over a very long period of time. Nevertheless, advances were being made in every other field. However, the invention of the new decimal place-value system was so significant a breakthrough that it permitted us to make the leap from the primitive agricultural society of 1500 to our space-age technology of 1970 in less than five hundred years.

Until the 1950s, most of our children were taught the decimal place-value system of arithmetic by rote. The only people who were interested in the "how" and "why" of it were the theoretical mathematicians who were intrigued by this ingenious tool and began to take it apart to see what made it tick. Many who took it apart, however, couldn't quite put it together again. Others came up with some discoveries about the system which could later be applied in the building of computers. However, none of this information was of any use to a child learning to master our arithmetic system. The system, as a practical tool, had been perfected by its users over the centuries.

Since most people find that arithmetic is all the mathematics they will ever need in their lifetimes, it is important to distinguish between arithmetic and mathematics. Arithmetic is a tool of economic man. It helps him deal with quantity, money, and measurement, all of which relate to practical everyday living. It is the practical tool which enables man to conduct the economic business of his life with speed, accuracy, and efficiency. Arithmetic, in other words, is much more allied to the world of commerce and industry than it is to the world of mathematical speculation and theorization. Arithmetic goes with those areas of interest in which computation plays an important part: economics, government, taxation, accounting, business management, etc. Mathematics, on the other hand, is more closely allied to physics, chemistry, engineering, astronomy, philosophy, metaphysics, etc. Thus, in developing a curriculum, it can be shown that arithmetic and mathematics lead toward two divergent paths of interest and activity. Of course, in our complex industrial civilization, all subject matter is interrelated. Reading, for example, is required in every field where the written word is used. But we teach reading as a separate and distinct skill. The same ought to be the case with arithmetic. Arithmetic is used in business as well as in mathematics and scientific calculation, and for this reason it should be taught as a separate and distinct discipline which the growing youngster can later adapt to his chosen field of interest.

What is arithmetic? Arithmetic is simply the art of counting. All arithmetic functions (addition, subtraction, multiplication, and division) are merely different ways of counting. In addition we count forward. In subtraction we count backward. In multiplication we count in multiples, which is merely a faster way of counting forward when dealing with great quantities. In division, the same principle is applied in the reverse direction. Until the invention of the Hindu place-value system all counting (or calculation and computing) was performed by way of abacuses and counting boards, in which quantitative units were represented by concrete objects such as pebbles or beads. Written numbers were used only to record totals.

The advent of the Hindu system marked a radical departure from this primitive concrete counting method. It permitted the accountant to do his calculating with the written symbols themselves. But to do this well, the accountant had to become proficient in the use of the abstract number symbols. The entire counting process had to be transferred from a pebble-manipulating process to a purely mental and written process—from the concrete to the abstract, and from the manipulation of things to the manipulation of symbols representing abstractions.

It is useful in dealing with the subject of abstraction (about which there is so much confusion in the minds of adults as well as children) to define our terms as accurately as possible. The words "abstract" and "abstraction" are used in so many different ways to mean so many different things, that people can easily be confused unless they understand the sense in which the term is used by a writer. One of the growing problems in communication today is the great proliferation of new words meaning new things, new words meaning old things, and old words being used in new ways. Dictionaries cannot keep up with the changes in spoken and written language, with the result that communication in complex areas of thought is on the verge of a breakdown. This is particularly true in modern pedagogy, where educators with different views and different axes to grind refuse to agree on definitions of terms. Sometimes the confusion is deliberately encouraged so that under the cover of verbal fog some educators can reap professional and financial benefits which clear verbal sunlight might make impossible.

In dealing with arithmetic in the context of the New Math it is very important to understand the terms being used, for the New Math has smothered arithmetic in a mass of so many new, complex, ill-defined words that both teacher and pupil can easily lose their way. Parents are so intimidated by the whole complicated approach, that they simply stay away from the New Math. Arithmetic wasn't always this coveted by the mathematician. In fact, arithmetic was of little interest to theoretical mathematicians while it was in its pebble-counting stage. The moment it began to use abstract symbols they took a closer look.

The verb *abstract* and the noun *abstraction* are derived from the Latin word *abstrahere*—to draw away from. This definition essentially describes how abstractions are created. The idea, or mental image, is drawn away from the concrete. It becomes an abstraction in our heads. How did this occur with numbers? The process probably started soon after man learned to use his vocal chords to create language. Undoubtedly man had mental images long before he began to speak. Those were his first abstractions. What he saw in outer reality could be seen in his head either in a dream or with his eyes closed. He also could hear the noises of outer reality in his dreams or by thinking of them. In other words, the ability to retain the impressions of the outer world which we receive through our senses is one of the properties of our nervous system. Those inner impressions, divorced from outer reality, were man's first experience with abstraction. It is probable that early man attributed to his dreams a reality they did not have, which is probably how much of our mythology originated.

Man's invention of spoken language was his first means of concretizing, by way of vocal sounds, elusive mental images and ideas, feelings and actions, so that he could get a firm grip on them. Spoken words became the first symbols to represent and thus communicate to others the abstractions in his head. By giving the ideas names he could anchor them down more securely, he could confirm their existence, and he could, in a sense, increase their reality. All of us know the power of words and man must have become aware of this phenomenon very early in his use of speech.

Also, spoken words enabled man to handle his abstractions with greater precision and thus to exert better mental control over them. The same was true of feelings and actions. By naming them he could better identify and understand them. To name something, it should be noted, is simply to designate it with a spoken expression. The common expression "What do you call it?" sums up the naming process.

Thus, our first spoken words were probably the names of persons and objects. These vocal expressions could only serve as a means of communication if more than one person agreed that the same sound meant the same thing. Language thus developed not only as a means of communication among people, but as a mental tool which individuals could use to develop their own powers of reasoning and understanding.

Did man start drawing before he started speaking? I doubt it. Drawing implies considerable cultural development. The most primitive aborigines speak but do not draw. In the evolution of abstractions, words like "mother" and "father" were probably first used as names, then generalized to mean all mothers and fathers. A word like "house" probably began as the name or designation of a specific living place and was later used to express the idea of a living place in general. The same can be said of "cat," "dog," "sister," "brother," and all the simple object names or nouns which infants learn when they start talking.

The important thing to understand here is the level of abstraction each word represents. The first level is a specific word applying to a specific object in reality. The relationship of the spoken symbol to the concrete is direct. The "drawing away" is only one step, or one level, away. On this level is also the generalized use of name words like "house," "cat," "dog" when referring to a specific entity. When we think of "feeding the cat" we are generally thinking of a specific cat. When the infant says "mama" and "dada" he is referring to specific people he knows, not to the concept of mother and father, or parenthood, or reproduction. When we use the word "sun" we are referring to that specific ball of yellow in the sky. Now, of course, we also speak of other suns in other solar systems.

Thus, in everyday speech we all use first-level abstractions in great abundance. The language of all children learning to speak is made up exclusively of first-level abstractions because they are the easiest to learn and have a direct connection to a concrete object. The child learns them by trial and error. He discovers which ones work and he is thrilled by the sense of discovery he experiences as he adds more and more words to his vocabulary. It should be noted that first-level abstractions include more than just names of objects. They include moral abstractions having to do with behavior as in such phrases as "good boy" and "bad boy," and action and feeling abstractions having to do with eating, drinking, walking, running, wanting, hurting, and the body functions.

The first arithmetic abstractions the child learns are those dealing with measurement: "big" and "little"; or quantity: "a little bit," "a lot," "more," "less," "one," "two," "three." Are these first-level abstractions? From what concrete objects are they drawn? "Big" and "little" are concepts having to do with comparisons. He learns the concepts of "big" and "little" because they are applied in concrete situations all around him: "big brother," "little baby," "small kitten," "big dog," "little puppy." "Big" and "little" become second-level abstractions because they are drawn away from first-level abstractions. Thus, the most elementary arithmetic concepts or ideas are second-level abstractions, twice removed from the concrete.

It is obvious that the transition to the use of second-level abstractions by a child represents a considerable intellectual development. He achieves this development all by himself through his own trial-and-error method of discovery. Obviously he is helped in this development by those around him whom he is trying to understand and imitate. It is also obvious that this transition to second-level abstractions is an uneven process. It takes time before one graduates from an understanding between "good boy," "bad boy," to "good behavior," "bad behavior," and the ethical concepts of good and evil.

The point we wish to make is that spoken language is a means of concretizing all abstraction and that there are different levels of abstraction, that is, ideas which may be once, twice, or three times removed from the original source in reality—including our own organic reality. It should be noted that no matter how complex our civilization becomes, we are always adding new first-level abstractions to our vocabulary: *rocket, missile, airplane, telephone, computer, light bulb, thermometer.* These are the easiest spoken abstractions to learn, no matter how complicated the origin of the words. They are simply names for concretes. While we are increasing the number of first-level abstractions in our vocabulary, we are also increasing the number of second- and third-level abstractions. Here is where we get into difficult problems of definition and here is the source of our present semantic confusion. The knowledge explosion has brought with it the abstraction explosion. In the field of mathematics the confusion is great. Much of it is due to the proliferation of undefined terms and the mixed use of alphabetic words with hieroglyphic symbols. The imposition of this mathematical confusion on arithmetic is causing many children great learning difficulties.

What is the origin of arithmetic abstraction? As primitive man's economic life grew, it became more and more necessary to keep track of larger quantities of goods and cattle. There is probably no concept more elusive than quantity. We can all conjure up mental images of two items, three items, four and five items. But very soon the mind simply cannot cope with larger quantities in this manner. The only way man could concretize and thus get a better grip on the idea of quantity was to give names to specific quantities. These names became known as numbers. Where did the names come from? Primitive man found that the easiest way to count (that is, to indicate exact quantity) was not by thinking about units in his head but by using his marvelous ten fingers. He gave each finger a name in a specific sequence: one through ten. Thus was born the first numbering system. Of course, we have no idea how many centuries it took primitive man to get this far. Obviously the idea of sequential position was closely related in his mind to that of total. He knew, for example, that he had five sons, and he knew the sequence of their birth. But for arithmetic it was easier to count fingers than children. In time, the names or numbers were "drawn away" from the actual fingers to become merely the names of quantities, each name easily remembered if recited in a specific sequence. *Sequential counting*, in other words, was the first aid to the memorization of number names.

Thus, language permitted man to concretize the elusive idea of quantity. By naming quantities man put a handle on something which does not exist in nature but exists in his head for his own use. As the quantities men dealt with increased, the number names over ten became compounds of the finger names, simplifying both the naming and the counting process. All this was done verbally, but because men were so used to calculating with the actual ten fingers, they easily learned to count in tens and to use a ten-base system. Permanent records were made with notches on wood, bundles of sticks, knots tied on a rope and other concrete devices.

In time man developed written symbols for the numbers, merely as a means of recording totals. All calculation, however, was performed by using such concretes as counting boards and abacuses.

With the advent of the alphabet, spoken numbers could also be written out phonetically. However, this was of no help from a computational point of view. Ideographic and hieroglyphic number symbols were still used for arithmetic notation and their use in computation was minimal. To clarify the distinction between ideographic and hieroglyphic numbers, we might say that line markings or a series of dots indicating units would be an ideographic way of writing numbers. They are the simplest graphic substitutes for concretes. The use of one symbol (a letter or a unique graphic design) to represent more than one unit would be a hieroglyphic number. All our Arabic numbers are hieroglyphics. The Greeks used alphabet letters as number hieroglyphics but failed to devise a place-value system whereby they could be used in computation. The Roman numbers were a mixture of ideographs and hieroglyphics, that is, a combination of unit marks and letters. This mixture of graphic concretes with symbols of abstractions proved to be quite unworkable for computation.

It is important to understand the complexity of the mental process involved in man's development of a counting system. When man first gave names to his fingers in order to count small quantities, he probably first conveyed the idea of a total by counting each unit until he reached the last one. Thus, to say "three" he had to say "one, two, three." Eventually he shortened the process so that whenever he said "three" everyone knew he meant the total, and not merely a position in sequence. What is significant is that each number came to represent not only a position in sequence and a specific quantity, but also the counting process itself which was implicit in the naming of a total. Thus, there was much more involved in the naming of quantities than one might suspect. By telescoping the counting process in each number over one, numbers took on an added abstract dimension. Thus, a number is a complex symbol with three important meanings: position in sequence (a first-level abstraction), quantity or total (a second-level abstraction), and the process whereby quantity is determined (another second-level abstraction). Remember, quantity or total does not exist in nature, and it is impossible for man to separate quantity from a counting process used to determine it. However, our minds, always eager to find shortcuts, combine the two concepts in one symbol with no difficulty whatever. It is the law of the conservation of energy which makes such efficient mental functioning possible.

All of this is true for the verbal numbers as well as our Arabic number symbols. All verbal numbers represent position in sequence, specific quantity, and the counting process. How do we know which meaning is intended? The context of our speech gives us clues. We say "He is third" or refer to "page 25" to indicate position. When we say "My house has ten windows" we indicate total as well as the fact that we counted the windows. When we say "I started with 100 dollars and have only 50 left" we are speaking about totals but the counting process is somewhat explicit. When we use our Arabic numbers, their meaning is further enhanced by the added dimension of place-value. For example, the number 123 represents the one-hundred-and-twenty-third position, the quantity of one-hundred-and-twenty-three units, and the counting process used to determine that total. But the position of the numerals in the number also determine their value in computation. It is quite possible to learn by sight-reading that the hieroglyphic 123 stands for one-hundred-and-twenty-three without knowing anything about place value. But if you want to compute with that number, you must know the meaning of place value.

Thus, our Arabic numbers have four meanings while our verbal numbers have only three. Is place value a first- or second-level abstraction? To answer that we must go back to the concrete from which place-value was drawn. That concrete was the counting board, in which each column represented ones, tens, hundreds, etc. We should imagine that this would make place value a first-level abstraction, except that in the counting board itself was incorporated the second-level abstraction of multiplication by tens in all the columns except the unit column. The zero was invented to indicate an empty column so that the accountant could indicate the different values of the digit 3 in 3, 30, 300. Thus, the zero was a first-level abstraction drawn from an empty column, but the digit 3 in 30 and 300 took on an additional second-level abstract meaning (i.e., multiplication by tens, hundreds, etc.). By transferring the second-level abstraction incorporated in the counting board directly to the numerals on paper, we can dispense with the counting board altogether, which is what happened when the Hindu system was adopted in Europe.

Thus, in place-value notation, each numeral assumes an additional second-level abstract meaning. This includes the zero. Although it started out as a first-level abstraction, it has taken on a second-level abstract dimension by becoming an integral part of a hieroglyphic number used in computation. Standing by itself, symbolizing the absence of quantity, the zero performs the unique function of concretizing the idea of nothing. In a way, it symbolizes man's awesome ability to deal with abstraction in daring, versatile, and ingenious ways.

What does all this mean? It means that behind the beautiful simplicity of our ten-symbol arithmetic system is a highly complex circuitry of abstraction. When we deal with numbers in arithmetic our minds are dealing simultaneously with four different meanings on two different levels of abstraction. This is no problem for an adult, but it is for a child of five or six. That is why arithmetic must be taught to children in a very orderly way—one thing at a time, to avoid the confusions that bad teaching can easily cause. Despite the complexity behind the system, once it is understood, the system easily suggests how it should be taught.

First, we must understand that arithmetic is nothing more than a tool for memory. Its prime function is to keep track of quantity and permit us to calculate quantity. The system's chief way of keeping track of quantity is by serial counting—that is, naming totals in sequence. Thus, the first step in teaching a child arithmetic is to teach him to count. As he is taught to count verbally, he is also taught to associate the number symbols with specific quantities. He does this by counting units. To do this he can use fingers or pennies. Since our ten-base system is derived from our fingers, our fingers are the most natural and convenient counting board the child can have. The use of these concrete units are only necessary up to ten, for beyond ten he is dealing with ten plus units of one, and all this can and should be done with the ten Arabic symbols he will be using for the rest of his life.

The instruction in this arithmetic primer has been arranged in "steps," rather than lessons, for the simple reason that the skills involved lend themselves more conveniently to this kind of an arrangement. A step may include concepts and skills which require a day to master, or a week, or a month before the child is ready to move on to the next step. It is important that the child master the material in one step before going on to the next. If the tutor finds the child becoming confused, one should then go back to the earliest point where the child was on solid ground and proceed again slowly in order to discover where the pupil missed the point. Do not proceed further until what was missed has been mastered.

It is suggested that the tutor read through the entire course of instruction before starting to teach. In that way the tutor will gain a better overall grasp of the methods and concepts used. It will be noted, for example, that once the child has learned the quantities the numbers 1 through 10 stand for, we use few problems with concretes for the child to solve. This is deliberate. Our arithmetic system is based on the manipulation of symbols—not apples, oranges, buttons, or other concrete units. These concretes hinder the development of automatic adding because they make the child think in terms of unit counting instead of adding, subtracting, multiplying or dividing with quantities or totals represented by number symbols.

Besides, most of these unit-counting problems are artifical, unreal, and boring. It isn't the numbers that bore children but the ridiculous items that textbooks attach to them. Children never have to perform arithmetic problems in real life with apples, balloons, cowboys, or peppermint sticks. They do count money, boxtops, trading stamps, and birthday candles. The child is not bored with arithmetic per se, which in itself is an interesting mental game. But he does get bored with meaningless "problems" which bind him to concrete unit counting and thereby retard his learning to add numbers automatically. The child wants to get on with the learning of arithmetic, not to determine how many monkeys you have left if you take two out of a cage holding five. No child will ever be confronted with the problem of counting monkeys and he doesn't know any adults who will have to count them either. It is such unreal problems which give children the feeling that arithmetic itself is unreal. Thus, avoid all counting problems which are clearly out of the realm of reality. Have the child count those things he would normally have to count or want to count, or those things which adults have to count but which lend themselves to instructional use by children. The child's weekly allowance or savings program might make an excellent basis for teaching elementary calculation.

Also, most of the arithmetic problems in today's primary textbooks are based on baby-think. It is assumed that a child must be surrounded by an array of baby chicks, circus clowns, and party hats in order to learn anything. We strongly disagree with this philosophy and suggest that the playful paraphernalia of babyhood not be permitted to smother the substance of learning. One should have respect for the child's intellectual capacity and curiosity, knowing how much he has learned by himself in his own way without the use of balloons and toy bugles. It should be noted that when children play games, they imitate adults not children. A little girl with a doll is trying to play the role of an adult mother to the best of her understanding. A little boy playing cowboys and Indians, or playing the role of a train motorman or jet pilot, is trying to act the part of a grown man.

Therefore, when we introduce the child to something as abstract as our arithmetic system, we must appeal to the budding adult in him, the part of him that has a curiosity about the outer world and wants to master basic skills. The way to instructional success is to teach one concept at a time, making that concept understandable and providing enough time and practice for the child to master what he is being taught.

All counting is a function of memory, and our arithmetic system depends on memorization for its most efficient use. Children enjoy memorizing if it is taught well, with good humor and patience, and as a learning challenge. Any child can be taught to memorize. Because memorizing is based on imitation, it is the easiest form of learning. That is why our arithmetic system lends itself so readily to the primary school curriculum. Counting itself is simply the memorization of additions by one. If a child can learn to count from one to a hundred, he can be taught to memorize the multiplication tables as well.

Remember, we are preparing the child now for what he will be doing twenty years from now. It is important that we keep this long-range view in mind and not let the colorful but mentally empty paraphernalia of childhood obscure the fact that sooner than we think this child will be an adult using the skills we taught him each and every day of his life.

Step 1: Find out the extent of the child's understanding of arithmetic concepts. Children of six will vary greatly in the amount of arithmetic knowledge they have picked up through their informal way of learning. Discuss the meaning of such words and phrases as *more, less, a little bit, a lot, many, few.* Find out the extent of the child's verbal counting ability. Some children of six can count as high as thirty. Others may not be able to count to ten. The ability to count, however, does not necessarily mean that the child understands the meaning of the numbers he recites. Therefore, ask the child if he understands what the numbers mean and why we use them. If he cannot give adequate answers, explain to him that numbers are used so that we can know *exactly* how many of anything we have or may need. For example, if we know that a candy bar we want costs five cents, then we know exactly how many pennies to get from mother. If we know that our friend Tim will be six on his next birthday, then we'll know exactly how many candles to put on his cake.

Establish the idea of exactness with number. Number tells us exactly how many as opposed to "a few" or "a lot."

Step 2: Write the Arabic numbers 1 through 10 on the blackboard and have the child learn to read them in proper sequence. Our purpose is to teach the child to identify the spoken number with its hieroglyphic counterpart. Also teach the child to write the numbers in conjunction with his handwriting lessons. Since we only use the Arabic symbols in arithmetic, the phonetically written numbers should be taught as part of reading instruction and not arithmetic. To include the spelled-out numbers at this point would detract from the arithmetic purpose of his learning.

As the child learns to identify a spoken number with a number symbol, he is also learning to count. Counting consists of naming quantities in their proper sequence, the constantly repeated sequence being the best aid to memory of the quantity names. Remember, counting is a pure task of memorization, and it is as difficult for a child to learn to count by ones as it is for an older child or many an adult to count by sevens or nines. If the child is given many opportunities to use his counting he will learn it quickly. Counting is perfected when it becomes completely automatic, that is, the child can repeat the numbers in sequence without hesitation and without thinking about them. This is a goal which only repeated counting makes possible.

Learning to count *over* ten is a lot easier than learning to count *to* ten, for numbers over ten are compounds of ten or tens with units repeated in the same original sequence of one through ten.

If the child can already count to ten and beyond, review his counting ability and teach him to associate the verbal number with the number symbol as far as he can count verbally. Then use your discretion in expanding his verbal and symbolic counting ability at this time. If there is good indication that he can easily learn to count to fifty or one hundred at this time, let him do so. With today's inflation, children are exposed to much larger sums in their purchases than were the children of twenty and thirty years ago. However, do not assume that because a child of six verbalizes a large number, he knows what it means. In general, it is not very valuable to the child to learn counting in the higher decades if he does not know the meaning of the numbers one through ten.

Step 3: In teaching the numbers one through ten, make sure the child understands the meaning of each number. He learns sequential position (the ordinal sense of the number) through recitation, but he must be taught the quantities each number stands for (the cardinal sense of the number). This can be done by the use of concrete units, such as fingers or pennies.

An easy way to teach the child the meaning of each number is to have him show you how many fingers make 3, or 5, or 7, etc. Also, with the use of pennies you can ask the child to give you 3 cents, 5 cents, 4 cents, 8 cents, etc., until the quantity which each number stands for is firmly established in his mind.

Fingers make an excellent unit counting board since they are always available. Also, since our numbering system is based on finger counting, the correlation between number symbol and the concrete units is perfect. The ten fingers provide an excellent concrete reference for the ten-base system. They also provide an easy, visible proof of the fact that 5 and 5 are 10, and that 2 times 5 are 10. These number relationships are so solidly based on our own physical reality that they become the most useful reference points the child can have in the entire number system. Also, later, in unit counting over ten, the child will learn through the spoken number, number symbol, and his fingers that eleven is ten-plus-one, twelve is ten-plus-two, etc. The compound names suggest the addition process in terms of ten. Therefore, the use of the fingers makes an excellent introduction to the ten-base system.

It should be noted that elementary mathematics textbooks refer to the ordinal number and cardinal number when referring respectively to a number's sequential position or its quantity. We find the use of such rarefied technical terms more of a hindrance to the understanding of arithmetic than a help. Would it not be more appropriate to identify the intended meaning of a number by referring to its "positional sense" or "quantity sense"? This would make arithmetic more understandable for both teacher and pupil. Positional sense need not be confused with place value, which comes into the picture much later.

Step 4: After the child has learned to count to ten and knows the quantities each number stands for, you will want to teach him to count with the symbols rather than units. You can start doing this by demonstrating that each number symbol in sequence represents one unit more than the preceding number. Thus he learns the following additions by one, which you can write out on the blackboard, explaining the meaning of the plus and equal signs. You can do the first two alone, then enlist the child's assistance in figuring out the totals of the rest.

$$1 + 1 = 2$$

$$2 + 1 = 3$$

$$3 + 1 = 4$$

$$4 + 1 = 5$$

$$5 + 1 = 6$$

$$6 + 1 = 7$$

$$7 + 1 = 8$$

$$8 + 1 = 9$$

$$9 + 1 = 10$$

Go over these until the child knows them in any order. Ask him: "What is 5 plus 1, 8 plus 1, 2 plus 1?" etc. This will reinforce his understanding of the number quantities as well as the meaning of sequential counting.

Step 5: You can show the child the same arithmetic facts learned in Step 4 in another way so that he can appreciate the convenience of our number symbols:

$$1 + 1 = 2$$

$$1 + 1 + 1 = 3$$

$$1 + 1 + 1 + 1 = 4$$

$$1 + 1 + 1 + 1 + 1 = 5$$

$$1 + 1 + 1 + 1 + 1 + 1 = 6$$

$$1 + 1 + 1 + 1 + 1 + 1 + 1 = 7$$

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 8$$

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 9$$

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 10$$

Such a demonstration will convince the child how much time and energy he can save by using the quantity symbols. Children are always happy when they find shortcuts. Here you show him how to eliminate unit counting by the use of number symbols, symbols that stand for quantities.

Step 6: To further strengthen the child's ability to use the number symbols in calculation rather than to count units, teach the following additions by showing how units can be combined in number symbols more than one, or, to put it in the reverse, how numbers are used to represent more than one unit. You can demonstrate the process on the blackboard by encircling the units represented by the numbers in the additions. Take as much time as is needed to teach these additions. Explain each step as you go. Note that the child is to be taught that 1 + 2 = 3 as well as 2 + 1 = 3, etc. Children do not automatically assume that because 1 + 2 = 3, that 2 + 1 = 3. They have to be shown it.

$1 + 1 = 2$

$$\begin{array}{r} 1 \\ \underline{1} \\ 2 \end{array}$$

$1 + 1 + 1 = 3$

$1 + 2 = 3$

$2 + 1 = 3$

$$\begin{array}{r} 1 \\ \underline{2} \\ 3 \end{array} \quad \begin{array}{r} 2 \\ \underline{1} \\ 3 \end{array}$$

$1 + 1 + 1 + 1 = 4$

$1 + 3 = 4$

$2 + 2 = 4$

$3 + 1 = 4$

$$\begin{array}{r} 1 \\ \underline{3} \\ 4 \end{array} \quad \begin{array}{r} 2 \\ \underline{2} \\ 4 \end{array} \quad \begin{array}{r} 3 \\ \underline{1} \\ 4 \end{array}$$

1 + 1 + 1 + 1 + 1 = 5

1 + 4 = 5

2 + 3 = 5

3 + 2 = 5

4 + 1 = 5

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ \hline 5 & 5 & 5 & 5 \end{array}$$

1 + 1 + 1 + 1 + 1 + 1 = 6

1 + 5 = 6

2 + 4 = 6

3 + 3 = 6

4 + 2 + 6

5 + 1 = 6

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \\ \hline 6 & 6 & 6 & 6 & 6 \end{array}$$

1 + 1 + 1 + 1 + 1 + 1 + 1 = 7

1 + 6 = 7

2 + 5 = 7

3 + 4 = 7

4 + 3 = 7

5 + 2 = 7

6 + 1 = 7

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ \hline 7 & 7 & 7 & 7 & 7 & 7 \end{array}$$

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 8$$

$$1 + 7 = 8$$

$$2 + 6 = 8$$

$$3 + 5 = 8$$

$$4 + 4 = 8$$

$$5 + 3 = 8$$

$$6 + 2 = 8$$

$$7 + 1 = 8$$

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \hline 8 & 8 & 8 & 8 & 8 & 8 & 8 \end{array}$$

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 9$$

$$1 + 8 = 9$$

$$2 + 7 = 9$$

$$3 + 6 = 9$$

$$4 + 5 = 9$$

$$5 + 4 = 9$$

$$6 + 3 = 9$$

$$7 + 2 = 9$$

$$8 + 1 = 9$$

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \hline 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \end{array}$$

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 10$$

$$1 + 9 = 10$$

$$2 + 8 = 10$$

$$3 + 7 = 10$$

$$4 + 6 = 10$$

$$5 + 5 = 10$$

$$6 + 4 = 10$$

$$7 + 3 = 10$$

$$8 + 2 = 10$$

$$9 + 1 = 10$$

$$\begin{array}{r} 1 \\ \underline{9} \\ 10 \end{array} \quad \begin{array}{r} 2 \\ \underline{8} \\ 10 \end{array} \quad \begin{array}{r} 3 \\ \underline{7} \\ 10 \end{array} \quad \begin{array}{r} 4 \\ \underline{6} \\ 10 \end{array} \quad \begin{array}{r} 5 \\ \underline{5} \\ 10 \end{array} \quad \begin{array}{r} 6 \\ \underline{4} \\ 10 \end{array} \quad \begin{array}{r} 7 \\ \underline{3} \\ 10 \end{array} \quad \begin{array}{r} 8 \\ \underline{2} \\ 10 \end{array} \quad \begin{array}{r} 9 \\ \underline{1} \\ 10 \end{array}$$

Note how these unit-grouping exercises strengthen the child's understanding of the numbers and the quantity of units each number stands for. He learns the proofs of these additions by being shown how the units are combined into numbers. Now show the child how these addition facts can be arranged in a very useful addition table which the child can utilize in memorizing the addition facts. Memorization is the only way to automatic addition. If the child does not memorize the addition totals, he will be unit counting indefinitely, and this will hinder or retard his mastery of arithmetic.

	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	
2	3	4	5	6	7	8	9	10		
3	4	5	6	7	8	9	10			
4	5	6	7	8	9	10				
5	6	7	8	9	10					
6	7	8	9	10						
7	8	9	10							
8	9	10								
9	10									
10										

Step 7: The additions in the table can also be put into column form. Write them out as follows so that the child can see the pattern:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \underline{3} & \underline{3} & \underline{3} & \underline{3} & \underline{3} & \underline{3} & \underline{3} \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

$$\begin{array}{rrrrrr} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ \hline 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

$$\begin{array}{rrrrr} 1 & 2 & 3 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \end{array}$$

$$\begin{array}{rrrr} 1 & 2 & 3 & 4 \\ 6 & 6 & 6 & 6 \\ \hline 7 & 8 & 9 & 10 \end{array}$$

$$\begin{array}{rrr} 1 & 2 & 3 \\ 7 & 7 & 7 \\ \hline 8 & 9 & 10 \end{array}$$

$$\begin{array}{rr} 1 & 2 \\ 8 & 8 \\ \hline 9 & 10 \end{array}$$

$$\begin{array}{r} 1 \\ 9 \\ \hline 10 \end{array}$$

Have the child read off these additions aloud repeating "one plus one is two, two plus one is three," etc., until he reaches "one plus nine is ten." Repeating the additions in sequential patterns helps memory. Just as it is easy to remember poems that rhyme, so it is easier to remember arithmetic facts when they are learned in progressive order. The child should articulate these additions so that the sounds of the combinations become familiar to him. You can also write out the same addition sequences on the blackboard without the sums and see how well he adds them up. You can later test him on how well he has learned these additions by giving him random addition combinations by flash card. Some he will know cold if they are easy, such as 5 plus 5 is ten. Others will be remembered by their position in sequence or relationship to other combinations, while yet others he will figure out by unit counting in his head or on his fingers. The optimum goal is to get him to know 4 plus 5 as automatically as he knows 1 plus 1, so that there is no need to unit count. Unit counting hinders speedy, effortless calculation, and some unit-counting habits picked up in early instruction can persist throughout one's life, constantly hindering easy calculation. If the child is permitted to rely on unit counting, he will not make the effort needed to memorize the addition facts.

Most modern textbooks give the child the kind of arithmetic problems that encourage unit counting in addition, negating the unique advantages that our arithmetic system has as a tool for memory. The instruction in this book excludes such teaching. Therefore, once the addition facts of numbers 1 through 10 are proven by demonstrations of unit counting, no further unit counting should be permitted in adding quantities over 1. The reason why we stress this is that the hardest habits to break are those learned by children in the early grades where they establish their ways of doing things. This goes for bad habits as well as good. It makes no sense to let a bad habit become established in the hope that it will eventually be replaced by a good one. Chances are that it won't. Thus, care should be taken to make sure that the child establishes good habits from the beginning. This can be done by being aware of the child's thinking methods and how he performs the additions we give him.

We have devised the instruction in this book to make it as easy as possible to establish good habits from the start and difficult to establish bad ones. However, no book of instruction is better than the teacher using it. Thus, it is up to the teacher to put the book's methodology into practice.

Step 8: By now the child will have spent about a year getting acquainted with numbers and learning his first forty-five addition facts. If he cannot as yet count to 100, teach him to do so. If he learns this much easily, proceed to teach him to count to 500 or 1000, depending on his ability. Since the hundreds repeat all the decade counting already learned, they will reinforce the child's counting ability from 1 through 100. In any case, he should have little trouble learning to count to 100 since he can easily discern the sequential pattern of 1 through 10 repeated in each line of tens. *At this point we are interested merely in getting the child to associate the verbal number with its hieroglyphic counterpart.*

Test his ability to name numbers from 1 to 100 by showing them to him at random on flash cards. Practice with higher numbers if you have taught him to count beyond 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Step 9: Teach the rest of the additions with numbers to ten. Use the addition groupings in Step 10 for drill purposes.

	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	12
3	4	5	6	7	8	9	10	11	12	13
4	5	6	7	8	9	10	11	12	13	14
5	6	7	8	9	10	11	12	13	14	15
6	7	8	9	10	11	12	13	14	15	16
7	8	9	10	11	12	13	14	15	16	17
8	9	10	11	12	13	14	15	16	17	18
9	10	11	12	13	14	15	16	17	18	19
10	11	12	13	14	15	16	17	18	19	20

Step 10: Drill the child on these additions in the same manner used in Step 7.

$$\begin{array}{r} 10 \\ 1 \\ \hline 11 \end{array} \quad \begin{array}{r} 10 \\ 2 \\ \hline 12 \end{array} \quad \begin{array}{r} 10 \\ 3 \\ \hline 13 \end{array} \quad \begin{array}{r} 10 \\ 4 \\ \hline 14 \end{array} \quad \begin{array}{r} 10 \\ 5 \\ \hline 15 \end{array} \quad \begin{array}{r} 10 \\ 6 \\ \hline 16 \end{array} \quad \begin{array}{r} 10 \\ 7 \\ \hline 17 \end{array} \quad \begin{array}{r} 10 \\ 8 \\ \hline 18 \end{array} \quad \begin{array}{r} 10 \\ 9 \\ \hline 19 \end{array} \quad \begin{array}{r} 10 \\ 10 \\ \hline 20 \end{array}$$

$$\begin{array}{r} 9 \\ 2 \\ \hline 11 \end{array} \quad \begin{array}{r} 9 \\ 3 \\ \hline 12 \end{array} \quad \begin{array}{r} 9 \\ 4 \\ \hline 13 \end{array} \quad \begin{array}{r} 9 \\ 5 \\ \hline 14 \end{array} \quad \begin{array}{r} 9 \\ 6 \\ \hline 15 \end{array} \quad \begin{array}{r} 9 \\ 7 \\ \hline 16 \end{array} \quad \begin{array}{r} 9 \\ 8 \\ \hline 17 \end{array} \quad \begin{array}{r} 9 \\ 9 \\ \hline 18 \end{array} \quad \begin{array}{r} 9 \\ 10 \\ \hline 19 \end{array}$$

$$\begin{array}{r} 8 \\ 3 \\ \hline 11 \end{array} \quad \begin{array}{r} 8 \\ 4 \\ \hline 12 \end{array} \quad \begin{array}{r} 8 \\ 5 \\ \hline 13 \end{array} \quad \begin{array}{r} 8 \\ 6 \\ \hline 14 \end{array} \quad \begin{array}{r} 8 \\ 7 \\ \hline 15 \end{array} \quad \begin{array}{r} 8 \\ 8 \\ \hline 16 \end{array} \quad \begin{array}{r} 8 \\ 9 \\ \hline 17 \end{array} \quad \begin{array}{r} 8 \\ 10 \\ \hline 18 \end{array}$$

$$\begin{array}{r} 7 \\ 4 \\ \hline 11 \end{array} \quad \begin{array}{r} 7 \\ 5 \\ \hline 12 \end{array} \quad \begin{array}{r} 7 \\ 6 \\ \hline 13 \end{array} \quad \begin{array}{r} 7 \\ 7 \\ \hline 14 \end{array} \quad \begin{array}{r} 7 \\ 8 \\ \hline 15 \end{array} \quad \begin{array}{r} 7 \\ 9 \\ \hline 16 \end{array} \quad \begin{array}{r} 7 \\ 10 \\ \hline 17 \end{array}$$

$$\begin{array}{cccccc} 6 & 6 & 6 & 6 & 6 & 6 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 & 16 \end{array}$$

$$\begin{array}{ccccc} 5 & 5 & 5 & 5 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \end{array}$$

$$\begin{array}{cccc} 4 & 4 & 4 & 4 \\ 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 \end{array}$$

$$\begin{array}{ccc} 3 & 3 & 3 \\ 8 & 9 & 10 \\ \hline 11 & 12 & 13 \end{array}$$

$$\begin{array}{cc} 2 & 2 \\ 9 & 10 \\ \hline 11 & 12 \end{array}$$

$$\begin{array}{c} 1 \\ 10 \\ \hline 11 \end{array}$$

Here are the same additions arranged in another pattern to facilitate memorization:

$$\begin{array}{cccccccccc} 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 \end{array}$$

$$\begin{array}{ccccccccc} 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 \end{array}$$

$$\begin{array}{cccccccc} 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 \end{array}$$

$$\begin{array}{ccccccc} 10 & 9 & 8 & 7 & 6 & 5 & 4 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 14 & 14 & 14 & 14 & 14 & 14 & 14 \end{array}$$

10	9	8	7	6	5
5	6	7	8	9	10
15	15	15	15	15	15

10	9	8	7	6
6	7	8	9	10
16	16	16	16	16

10	9	8	7
7	8	9	10
17	17	17	17

10	9	8
8	9	10
18	18	18

10	9
9	10
19	19

10
10
20

To test the child's mastery of the addition facts, give the child random additions to perform. Note the ones that make him hesitate and note the ones he can add automatically. Make a list of the ones he is unsure of and drill him in them. Take plenty of time and be patient. Let him see and hear the more difficult combinations over and over again with the correct sums. In time they will become as easy to remember as 2 plus 2 is 4. Speed in response is important, because it permits us to detect unit counting. Make such speed drills as enjoyable and as pleasant as possible, telling the child that with sufficient practice he'll be able to perform perfectly. A good way to drill is to put addition combinations on flash cards. Such drills should be conducted in short spurts rather than in long tiresome sessions. But they should be repeated over a period of time until performance is perfect. We can all count from 1 to 100 at top speed without thinking because we have done it so often. Automatic knowledge is acquired by going over the same thing often enough so that a "path" in our minds is created. We acquire a great deal of automatic knowledge without being aware that we are doing so. However, with a system as complex yet compact and organized as arithmetic, such knowledge is best acquired through a systematic, deliberate approach. In the long run it prevents poor arithmetic habits from developing and saves the student time and effort all his life.

Step 11: Review counting and advance the child's counting capability to 1,000. Again, the purpose of the instruction is to teach the child to associate the verbal number with its proper hieroglyphic counterpart. This is done by teaching the child to see and hear the basic pattern of 1 through 10 in each decade of the hundreds. The child learns to count by remembering the recurring patterns in both verbal and symbolic numbers.

Step 12: Subtraction. Pose some simple subtraction problems to the child so that he can understand what subtraction is. Explain to him that while adding makes the quantity we started with more, subtraction makes that quantity less. As an example, ask the child if he had ten pennies and spent six, how many he would have left. He can use his fingers to see the subtraction process in concretes, or use pennies themselves. If he takes away three from five, how many does he have left? When the child firmly grasps the concept of subtraction in terms of units and in terms of a smaller quantity being taken away from a larger quantity, then give him the following simple subtractions in number symbols, explaining the meaning of the minus sign:

2 - 1 = ?	4 - 2 = ?	4 - 3 = ?
3 - 2 = ?	3 - 1 = ?	4 - 1 = ?

Explain that in addition we count forward. In subtraction, we really count backward.

Step 13: Teach the child to count backward from 20 to 1. Then demonstrate with the following how counting backward is really subtraction by ones, as counting forward is addition by ones:

9 + 1 = 10	10 - 1 = 9
8 + 1 = 9	9 - 1 = 8
7 + 1 = 8	8 - 1 = 7
6 + 1 = 7	7 - 1 = 6
5 + 1 = 6	6 - 1 = 5
4 + 1 = 5	5 - 1 = 4
3 + 1 = 4	4 - 1 = 3
2 + 1 = 3	3 - 1 = 2
1 + 1 = 2	2 - 1 = 1

Have the child learn the following subtractions with their addition counterparts. Show how subtraction can be checked by adding the difference (or remainder) to the subtractor (subtrahend):

8 + 2 = 10	10 - 2 = 8	7 + 3 = 10	10 - 3 = 7
7 + 2 = 9	9 - 2 = 7	6 + 3 = 9	9 - 3 = 6
6 + 2 = 8	8 - 2 = 6	5 + 3 = 8	8 - 3 = 5
5 + 2 = 7	7 - 2 = 5	4 + 3 = 7	7 - 3 = 4
4 + 2 = 6	6 - 2 = 4	3 + 3 = 6	6 - 3 = 3
3 + 2 = 5	5 - 2 = 3	2 + 3 = 5	5 - 3 = 2
2 + 2 = 4	4 - 2 = 2	1 + 3 = 4	4 - 3 = 1
1 + 2 = 3	3 - 2 = 1		

6 + 4 = 10 10 - 4 = 6

5 + 4 = 9 9 - 4 = 5

4 + 4 = 8 8 - 4 = 4

3 + 4 = 7 7 - 4 = 3

2 + 4 = 6 6 - 4 = 2

1 + 4 = 5 5 - 4 = 1

5 + 5 = 10 10 - 5 = 5

4 + 5 = 9 9 - 5 = 4

3 + 5 = 8 8 - 5 = 3

2 + 5 = 7 7 - 5 = 2

1 + 5 = 6 6 - 5 = 1

4 + 6 = 10 10 - 6 = 4

3 + 6 = 9 9 - 6 = 3

2 + 6 = 8 8 - 6 = 2

1 + 6 = 7 7 - 6 = 1

3 + 7 = 10 10 - 7 = 3

2 + 7 = 9 9 - 7 = 2

1 + 7 = 8 8 - 7 = 1

2 + 8 = 10 10 - 8 = 2

1 + 8 = 9 9 - 8 = 1

1 + 9 = 10 10 - 9 = 1

Through repeated recitations of these additions and subtractions, the child will not fail to detect the consistent counting patterns in our arithmetic system. In being exposed to it, his mind is bound to capture some of its logic. In our introduction we stated that our goal was to both teach arithmetic and have the child understand it. By detecting the consistency of the patterns, he will begin to understand the logic behind the system. The discovery of a logical consistency in a set of abstract symbols will be an exciting discovery for the child. It is more important for him to learn this than to solve unreal arithmetic problems which give him practice in unit counting, much to his own detriment.

Step 13a: The symbol for zero. Ask the child to give the answers to the following subtractions:

$1 - 1 = ?$ $\qquad$ $5 - 5 = ?$ $\qquad$ $10 - 10 = ?$

If he cannot give the correct answers, ask the child how much would be left if you took one away from one, or five from five, or ten from ten. If he answers "nothing is left," tell him that he is correct and that we use the zero, 0, to represent nothing. For example, when a baseball team scores nothing, we put down zero as the score. Give the child some additions and subtractions with zero, such as:

$1 + 0 = 1$ $\qquad$ $5 + 0 = 5$ $\qquad$ $10 + 0 = 10$

$1 - 0 = 1$ $\qquad$ $5 - 0 = 5$ $\qquad$ $10 - 0 = 10$

Tell the pupil that if he has any trouble adding or subtracting with zero to simply remember that zero stands for nothing, and that nothing added to something does not increase the something, and that nothing taken away or subtracted from something does not decrease the something. Thus, five plus nothing remains five, and five minus nothing remains five.

Step 14: Subtraction table. Show the child how to use the table in memorizing subtractions from 0 through 10.

0	1	2	3	4	5	6	7	8	9	10
1	0	1	2	3	4	5	6	7	8	9
2		0	1	2	3	4	5	6	7	8
3			0	1	2	3	4	5	6	7
4				0	1	2	3	4	5	6
5					0	1	2	3	4	5
6						0	1	2	3	4
7							0	1	2	3
8								0	1	2
9									0	1
10										0

Step 15: The child may find it easier to memorize the following subtractions from sums up to 19 in this form. Show him how he can check his subtractions by adding the remainder and the subtractor, which should equal the sum subtracted from.

1	2	3	4	5	6	7	8	9	10
-0	-1	-2	-3	-4	-5	-6	-7	-8	-9
1	1	1	1	1	1	1	1	1	1

2	3	4	5	6	7	8	9	10	11
-0	-1	-2	-3	-4	-5	-6	-7	-8	-9
2	2	2	2	2	2	2	2	2	2

3	4	5	6	7	8	9	10	11	12
-0	-1	-2	-3	-4	-5	-6	-7	-8	-9
3	3	3	3	3	3	3	3	3	3

4	5	6	7	8	9	10	11	12	13
-0	-1	-2	-3	-4	-5	-6	-7	-8	-9
4	4	4	4	4	4	4	4	4	4

5	6	7	8	9	10	11	12	13	14
-0	-1	-2	-3	-4	-5	-6	-7	-8	-9
5	5	5	5	5	5	5	5	5	5

6	7	8	9	10	11	12	13	14	15
-0	-1	-2	-3	-4	-5	-6	-7	-8	-9
6	6	6	6	6	6	6	6	6	6

7	8	9	10	11	12	13	14	15	16
-0	-1	-2	-3	-4	-5	-6	-7	-8	-9
7	7	7	7	7	7	7	7	7	7

8	9	10	11	12	13	14	15	16	17
-0	-1	-2	-3	-4	-5	-6	-7	-8	-9
8	8	8	8	8	8	8	8	8	8

9	10	11	12	13	14	15	16	17	18
-0	-1	-2	-3	-4	-5	-6	-7	-8	-9
9	9	9	9	9	9	9	9	9	9

10	11	12	13	14	15	16	17	18	19
-0	-1	-2	-3	-4	-5	-6	-7	-8	-9
10	10	10	10	10	10	10	10	10	10

Step 16: Drill subtractions with random combinations taken from the subtraction tables.

Step 17: Drill the fundamental addition and subtraction facts by mixing random combinations from both the addition and subtraction tables.

Step 18: Elementary addition involves more than simply learning the fundamental addition facts. Further techniques are basic to development of addition skills. These include column addition, higher decade addition—which is needed in column addition—and carrying. In teaching column addition we start off with three single-digit numbers which add up to not over 10. For example, we give the child 2, 3, and 4 to add in a column. He first adds the 2 and 3, which gives him 5. Then he adds 5 and 4 to give him the final sum of 9. What is significant in this process is that in adding 2 and 3 the child sees both numbers. But in adding 5 and 4, the 5 is not seen on paper but is held as an idea or image in his head, to which he adds the 4. The process of adding a written number to an invisible or mental number is a more difficult computational task for the child to perform. So we start him off with very simple addition columns, all of which add up to not more than 10. Lots of practice in adding these columns will accustom the child to adding mental numbers to written numbers.

1	1	1	2	1	1	3	1	1	4	1	1	5	1	1
1	1	2	1	1	3	1	1	4	1	1	5	1	1	6
1	2	1	1	3	1	1	4	1	1	5	1	1	6	1

6	1	1	7	1	1	8	1	2	2	1	1	2	2	3
1	1	7	1	1	8	1	2	1	2	2	3	1	3	1
1	7	1	1	8	1	1	2	2	1	3	2	3	1	2

3	1	1	2	2	4	4	1	1	2	2	5	5	1	1
2	2	4	1	4	1	2	2	5	1	5	1	2	2	6
1	4	2	4	1	2	1	5	2	5	1	2	1	6	2

2	2	6	6	1	1	2	2	7	7	1	3	3	1	1
1	6	1	2	2	7	1	7	1	2	3	1	3	3	4
6	1	2	1	7	2	7	1	2	1	3	3	1	4	3

3	3	4	4	1	1	3	3	5	5	1	1	3	3	6
1	4	1	3	3	5	1	5	1	3	3	6	1	6	1
4	1	3	1	5	3	5	1	3	1	6	3	6	1	3

6	1	4	4	1	1	4	4	5	5	2	2	2	3	2
3	4	1	4	4	5	1	5	1	4	2	2	3	2	2
1	4	4	1	5	4	5	1	4	1	2	3	2	2	4

2	4	2	2	5	2	2	6	2	3	3	2	2	3	3
4	2	2	5	2	2	6	2	3	2	3	3	4	2	4
2	2	5	2	2	6	2	2	3	3	2	4	3	4	2

4	4	2	2	3	3	5	5	2	4	4	3	3	3	4
2	3	3	5	2	5	2	3	4	2	4	3	3	4	3
3	2	5	3	5	2	3	2	4	4	2	3	4	3	3

Step 19: The next series of column additions to be learned are those in which the three one-digit numbers add up to sums over 10. In the additions below, the sum of the first two numbers in the column do not exceed 9. In learning to do these additions, make sure that the child always adds from the top down. This is the direction in which the numbers are written. It is the direction in which he should add them. Later on when he is well habituated to downward adding and has learned to add one-digit numbers to two-digit numbers in what is known as higher decade addition, you might introduce him to upward adding merely as a check on the accuracy of his downward addition.

1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	2	3	3	3	4	4	4	4	5	5	5	5
9	8	9	7	8	9	6	7	8	9	5	6	7	8

1	1	1	1	1	1	1	1	1	1	2	2	2	2
5	6	6	6	6	7	7	7	8	8	2	2	2	3
9	6	7	8	9	7	8	9	8	9	7	8	9	6

2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	4	4	4	4	4	5	5	5	5	5	6
7	8	9	5	6	7	8	9	5	6	7	8	9	6

2	2	2	2	2	2	3	3	3	3	3	3	3	3
6	6	6	7	7	7	3	3	3	3	3	4	4	4
7	8	9	7	8	9	5	6	7	8	9	4	5	6

3	3	3	3	3	3	3	3	3	3	3	3	4	4
4	4	4	5	5	5	5	5	6	6	6	6	4	4
7	8	9	5	6	7	8	9	6	7	8	9	4	5

4	4	4	4	4	4	4	4	4
4	4	4	4	5	5	5	5	5
6	7	8	9	5	6	7	8	9

Step 20: Higher decade addition. The purpose of this skill is to enable the child to add one-digit numbers to two-digit numbers mentally. It is a skill needed in order to become proficient in column addition. It is also useful in multiplication carrying. Basically it is a counting skill made automatic by much practice. Such exercises include the following kinds of additions in which we demonstrate how the fundamental addition facts relate to higher decade addition:

$$\begin{array}{rrrr} 3 & 13 & 23 & 33 \\ 4 & 4 & 4 & 4 \\ \hline 7 & 17 & 27 & 37 \end{array} \qquad \begin{array}{rrrr} 7 & 47 & 67 & 87 \\ 2 & 2 & 2 & 2 \\ \hline 9 & 49 & 69 & 89 \end{array}$$

$$\begin{array}{rrrr} 4 & 14 & 24 & 34 \\ 5 & 5 & 5 & 5 \\ \hline 9 & 19 & 29 & 39 \end{array} \qquad \begin{array}{rrrr} 6 & 36 & 56 & 76 \\ 3 & 3 & 3 & 3 \\ \hline 9 & 39 & 59 & 79 \end{array}$$

$$\begin{array}{rrrr} 2 & 12 & 22 & 32 \\ 8 & 8 & 8 & 8 \\ \hline 10 & 20 & 30 & 40 \end{array} \qquad \begin{array}{rrrr} 3 & 13 & 43 & 93 \\ 5 & 5 & 5 & 5 \\ \hline 8 & 18 & 48 & 98 \end{array}$$

In all such higher decade additions, the reverses should also be practiced. That is, 87 plus 2 should also be practiced with 2 plus 87.

Step 21: Bridging in higher decade addition. What is important to note in higher decade addition is that the principle of learning them is the same applied to the basic addition facts. They are to be learned so that the pupil need not unit count in his head. Obviously, the key to proficient higher decade addition is a flawless, automatic knowledge of the basic addition facts. The skill of "bridging," that is adding mentally from one decade to the next as in $28 + 7 = 35$, is more difficult for the child to master, but it can be done if taught systematically. Since higher decade addition should be learned as an automatic skill, an understanding of place value at this point is not necessary. The two-digit numbers are being used as single hieroglyphics designating specific quantities to which are added one-digit numbers designating specific quantities. In other words, the place value meaning of the two-digit number is irrelevant in this sort of adding. The verbal number and the recurring verbal and visual patterns in the higher decades are the keys to this kind of addition.

Place value only becomes a matter of importance when the child must start carrying when performing computation with two or more two-digit numbers. When we look at, say, the number 24 as a simple hieroglyphic, we see it as simply representing twenty-four single units. If we see it in terms of place value, we see it as 2 tens and 4 ones. It is easier to understand the process of carrying when we see the number in terms of place value. But in automatic counting, place value is only a help in that it provides certain visual and verbal number patterns which serve as effective aids to memory. But there is more to 24 than the fact that it is made up of 2 tens and 4 ones. It is also made up of two dozen, six fours, four sixes, three eights, eight threes. Thus, the place value meaning of 24 should be stressed when it is relevant to the arithmetic problem at hand.

Here are the addition exercises to teach bridging in higher decade addition:

9	19	29	39	49	59	69	79	89
2	2	2	2	2	2	2	2	2
11	21	31	41	51	61	71	81	91
8	18	28	38	48	58	68	78	88
3	3	3	3	3	3	3	3	3
11	21	31	41	51	61	71	81	91
7	17	27	37	47	57	67	77	87
4	4	4	4	4	4	4	4	4
11	21	31	41	51	61	71	81	91
6	16	26	36	46	56	66	76	86
5	5	5	5	5	5	5	5	5
11	21	31	41	51	61	71	81	91
5	15	25	35	45	55	65	75	85
6	6	6	6	6	6	6	6	6
11	21	31	41	51	61	71	81	91
4	14	24	34	44	54	64	74	84
7	7	7	7	7	7	7	7	7
11	21	31	41	51	61	71	81	91

3	13	23	33	43	53	63	73	83
8	8	8	8	8	8	8	8	8
11	21	31	41	51	61	71	81	91

2	12	22	32	42	52	62	72	82
9	9	9	9	9	9	9	9	9
11	21	31	41	51	61	71	81	91

9	19	29	39	49	59	69	79	89
3	3	3	3	3	3	3	3	3
12	22	32	42	52	62	72	82	92

8	18	28	38	48	58	68	78	88
4	4	4	4	4	4	4	4	4
12	22	32	42	52	62	72	82	92

7	17	27	37	47	57	67	77	87
5	5	5	5	5	5	5	5	5
12	22	32	42	52	62	72	82	92

6	16	26	36	46	56	66	76	86
6	6	6	6	6	6	6	6	6
12	22	32	42	52	62	72	82	92

5	15	25	35	45	55	65	75	85
7	7	7	7	7	7	7	7	7
12	22	32	42	52	62	72	82	92

4	14	24	34	44	54	64	74	84
8	8	8	8	8	8	8	8	8
12	22	32	42	52	62	72	82	92

3	13	23	33	43	53	63	73	83
9	9	9	9	9	9	9	9	9
12	22	32	42	52	62	72	82	92

9	19	29	39	49	59	69	79	89
4	4	4	4	4	4	4	4	4
13	23	33	43	53	63	73	83	93

8	18	28	38	48	58	68	78	88
5	5	5	5	5	5	5	5	5
13	23	33	43	53	63	73	83	93

7	17	27	37	47	57	67	77	87
6	6	6	6	6	6	6	6	6
13	23	33	43	53	63	73	83	93

6	16	26	36	46	56	66	76	86
7	7	7	7	7	7	7	7	7
13	23	33	43	53	63	73	83	93

5	15	25	35	45	55	65	75	85
8	8	8	8	8	8	8	8	8
13	23	33	43	53	63	73	83	93

4	14	24	34	44	54	64	74	84
9	9	9	9	9	9	9	9	9
13	23	33	43	53	63	73	83	93

9	19	29	39	49	59	69	79	89
5	5	5	5	5	5	5	5	5
14	24	34	44	54	64	74	84	94

8	18	28	38	48	58	68	78	88
6	6	6	6	6	6	6	6	6
14	24	34	44	54	64	74	84	94

7	17	27	37	47	57	67	77	87
7	7	7	7	7	7	7	7	7
14	24	34	44	54	64	74	84	94

6	16	26	36	46	56	66	76	86
8	8	8	8	8	8	8	8	8
14	24	34	44	54	64	74	84	94

5	15	25	35	45	55	65	75	85
9	9	9	9	9	9	9	9	9
14	24	34	44	54	64	74	84	94

9	19	29	39	49	59	69	79	89
6	6	6	6	6	6	6	6	6
15	25	35	45	55	65	75	85	95

8	18	28	38	48	58	68	78	88
7	7	7	7	7	7	7	7	7
15	25	35	45	55	65	75	85	95

7	17	27	37	47	57	67	77	87
8	8	8	8	8	8	8	8	8
15	25	35	45	55	65	75	85	95

6	16	26	36	46	56	66	76	86
9	9	9	9	9	9	9	9	9
15	25	35	45	55	65	75	85	95

9	19	29	39	49	59	69	79	89
7	7	7	7	7	7	7	7	7
16	26	36	46	56	66	76	86	96

8	18	28	38	48	58	68	78	88
8	8	8	8	8	8	8	8	8
16	26	36	46	56	66	76	86	96

7	17	27	37	47	57	67	77	87
9	9	9	9	9	9	9	9	9
16	26	36	46	56	66	76	86	96

9	19	29	39	49	59	69	79	89
8	8	8	8	8	8	8	8	8
17	27	37	47	57	67	77	87	97

8	18	28	38	48	58	68	78	88
9	9	9	9	9	9	9	9	9
17	27	37	47	57	67	77	87	97

9	19	29	39	49	59	69	79	89
9	9	9	9	9	9	9	9	9
18	28	38	48	58	68	78	88	98

After the pupil becomes familiar with these bridging patterns and sees the relationship between 9 plus 8 and 79 plus 8, you can arrange these additions in another pattern to aid memorization as in the following sample:

9	9	9	9	9	9	9	9
2	3	4	5	6	7	8	9
11	12	13	14	15	16	17	18

19	19	19	19	19	19	19	19
2	3	4	5	6	7	8	9
21	22	23	24	25	26	27	28

29	29	29	29	29	29	29	29
2	3	4	5	6	7	8	9
31	32	33	34	35	36	37	38

39	39	39	39	39	39	39	39
2	3	4	5	6	7	8	9
41	42	43	44	45	46	47	48

All these bridging additions should be studied and practiced in the context of the patterns so that the child forms in his mind the necessary number pattern aids to memory. Then you can flash random bridging additions on cards to see how well the pupil can perform them mentally. If there is much hesitation or obvious unit counting, find out which combinations cause the most trouble and practice them until the pupil can add them without difficulty. Mastery will require much practice, but mastery has its rewards. The child gains confidence in himself and in his ability to learn and use arithmetic. Memorization is the easiest form of learning, and arithmetic is basically a memorization system. The key aids to the use of the fundamental addition facts in higher decade addition are the recurring verbal and visual patterns. If more time is given to the more difficult combinations, they will be learned as thoroughly as the easier ones.

After the pupil has had sufficient practice in higher decade addition and bridging, give him sufficient column addition to put his skill to work. The tutor can prepare column additions which start off easy and get progressively more difficult, as in the following examples:

Higher decade combinations in the teens:

2	3	2	2	2	6	5	3	3
3	3	4	1	2	1	2	5	6
5	5	5	8	7	3	6	7	4
5	8	5	2	1	6	4	3	4

2	2	2	1	8	5	6	4	2
6	8	1	3	1	4	3	4	3
3	1	8	6	2	8	6	8	7
5	8	4	3	7	1	3	1	2

3	3	1	7	5	4	1	5	4
1	3	4	2	1	5	6	2	3
7	8	5	4	5	7	3	9	7
2	1	4	5	4	2	9	2	2

Higher decade combinations in the twenties:

8	2	7	9	6	5	7	8	6
2	8	8	3	6	5	4	3	8
5	4	2	3	6	5	6	9	2
8	9	4	6	3	8	7	5	5

7	8	5	7	6	4	6	5	9
3	1	6	8	7	6	5	7	8
9	7	2	2	6	9	7	7	3
2	6	7	8	7	4	6	1	9

7	8	4	6	4	4	3	9	6
7	8	6	4	9	8	9	6	9
3	3	8	4	1	2	6	3	3
6	8	7	7	8	6	2	9	4

Higher decade combinations in the twenties with five numbers:

8	9	6	7	9	8	4	7	5
4	2	6	4	6	3	7	7	8
3	5	7	2	2	1	7	3	1
7	4	4	9	9	6	7	3	8
3	8	2	5	3	2	2	6	4

3	8	6	8	3	7	9	5	9
7	3	5	3	2	6	9	4	8
2	8	7	1	9	3	1	6	2
8	3	3	9	7	8	6	5	5
4	3	4	6	6	3	3	8	2

7	8	4	6	4	4	9	4	8
7	8	6	4	9	8	3	8	7
3	3	8	4	1	2	1	5	1
6	8	7	7	8	6	8	6	3
2	1	3	2	6	5	7	4	9

More addition columns can be prepared by the tutor for the practice of higher decade additions, including some into the thirties.

Step 22: Place value. With the child now ready to start adding more than one two-digit number, we can introduce him to place value in order to give him an understanding of how these additions are performed. First point out how the number 10 was created by the addition of ten ones. Demonstrate how the ones can be added in column form. Then show how 20, 30, 40, 50, 60, 70, 80, 90, and 100 are created by the additions of 10s in the same way:

1									
1									
1									
1									
1									
1									
1									
1									
1									
1									
10	10	10	10	10	10	10	10	10	10
	10	10	10	10	10	10	10	10	10
	20	10	10	10	10	10	10	10	10
		30	10	10	10	10	10	10	10
			40	10	10	10	10	10	10
				50	10	10	10	10	10
					60	10	10	10	10
						70	10	10	10
							80	10	10
								90	10
									100

Explain that in a two-digit number, the digit on the right represents the ones or units column, while the digit on the left represents the tens column. Thus, the number 10 indicates that there are no ones and one ten. The number 20 indicates no ones and two tens, and so on. The number 12 indicates two ones and one ten. Give the child random two-digit numbers and see if he can identify how many ones and how many tens each digit stands for. Thus, in our decimal place-value system, each number of two digits tells us how many tens and ones it is made up of. This information is a built-in feature of our numbering system.

Then explain how in a three-digit number like 100, the third digit to the left stands for hundreds. Have the pupil read the following three-digit numbers and explain how many ones, tens, and hundreds each number has:

125 206 333 490 521 685 709 820 999

To further demonstrate the place-value concept and how we add numbers in place-value columns, show how the above three-digit numbers can be written out in the form of column additions as follows:

5		3		1	5			9
20	6	30	90	20	80	9	20	90
100	200	300	400	500	600	700	800	900
125	206	333	490	521	685	709	820	999

Explain that in our arithmetic system we add each column starting at the right and proceeding to the left.

Step 23: Give the pupil simple two- and three-digit combinations to add in which no column adds up to more than nine, as in the following sample:

13	21	74	52	60	25	44	53	40	27
12	18	23	25	28	24	11	26	22	61

49	120	302	625	550	921	256	137	471	765
50	142	140	321	42	33	103	601	404	233

Step 24: Carrying. If in adding combinations of two-digit numbers the sum of the tens column is over nine, we put down the ones digit of the sum in the ones column and "carry" to the tens column the tens digit. Then we add up the tens column to get its sum. For example:

	1	1
12	12	12
19	19	19
	1	31

	1	1
11	11	11
19	19	19
	0	30

Give the pupil carrying exercises with the additions below. You can devise as many as you need for practice purposes.

19	18	37	26	18	27	26	29	37
21	32	23	54	23	35	47	65	28

38	49	19
58	18	39

In this next group, the sums are over 100:

85	66	96	82	75	87	53	84	63
36	76	88	18	26	74	69	99	67

99	57
99	88

Step 25: Adding three-digit numbers with no carrying. The columns are added from right to left.

123	202	333	420	511	433	683	770	856	284
405	141	333	419	238	542	205	217	143	603

Step 26: Adding three-digit numbers with carrying in the tens column. Explain each step.

	1	1	1
119	119	119	119
24	24	24	24
	3	43	143

1	1	1	1
119	119	119	119
124	124	124	124
	3	43	243

Have the pupil practice with these additions and others you may devise:

238	427	209	109	336	288	175	637	288	347
142	534	609	208	424	607	315	236	409	235

Step 27: Adding three-digit numbers with carrying in the hundreds column. Explain each step:

		1	1
191	191	191	191
191	191	191	191
	2	82	382

		1	1
170	170	170	170
462	462	462	462
	2	32	632

Have the pupil practice with these additions below. Note that the sum of the tens column is less than ten.

284	373	365	457	571	563	222	775	286	620
194	184	362	352	336	282	192	164	182	294

Step 28: Adding three-digit numbers with carrying in the tens and hundreds columns. Explain each step:

	1	11	11
199	199	199	199
123	123	123	123
	2	22	322

	1	11	11
185	185	185	185
115	115	115	115
	0	00	300

Have the pupil practice with the following additions, to which can be added more. Make sure that the pupil places the carried digit on top of the proper column and writes it as legibly but not as large as the other digits in the numbers. The purpose of this is to distinguish the carried numbers from the addends, that is, the numbers being added.

166	167	158	383	298
266	177	468	417	289

256	375	296	488
465	457	147	152

Step 29: Adding three-digit numbers with carrying in the tens and hundreds columns adding up to sums over 999. Explain each step.

	1	11	11
899	899	899	899
101	101	101	101
	0	00	1000

Explain that in a four-digit number, the fourth digit to the left is the thousands column. Have the pupil practice with these additions:

956	874	548	832	639	751	388	473	927	654
655	357	453	878	363	659	756	847	996	889

Step 30: Subtraction with two-digit numbers in which borrowing is not needed. Provide more practice with additional combinations if needed:

```
 53    64    79    85    78    99
-21   -43   -45   -32   -50   -75
---   ---   ---   ---   ---   ---

 86    47    57    69    71
-61   -35   -43   -21   -40
---   ---   ---   ---   ---
```

Step 31: Subtraction terms.

```
 28   minuend (the sum subtracted from)
-13   subtractor (the amount subtracted)
---
 15   remainder (what's left)
```

In some textbooks the *subtractor* is called the *subtrahend* and the *remainder* is referred to as the *difference.*

Step 32: Borrowing in subtraction. The method used in this instruction is known by three different names: the take-away-carry method, the method of equal additions, or the borrow-and-payback method. It's most accurate description is as the method of equal additions. Thus, when the "ones" digit in the minuend is lower than the "ones" digit in the subtractor, we add ten to the "ones" digit in the minuend and one to the "tens" digit in the subtractor. Over the centuries this adding process has come to be known as "borrowing." The operation works as follows:

1. We start with this subtraction problem.

```
 72
-26
---
```

2. We "borrow" ten and add it to the 2 in the minuend. This makes it 12. We then subtract 6 from 12. We write the remainder of 6 under the "ones" column.

$$\begin{array}{r} 7\;{}^{1}2 \\ -2\;\;6 \\ \hline 6 \end{array}$$

3. But we must also add ten to the "tens" digit of the subtractor. We do this by adding 1, remembering that the digits in the "tens" column represent multiples of ten.

$$\begin{array}{r} 7\;{}^{1}2 \\ -{}^{1}2\;\;6 \\ \hline 6 \end{array}$$

4. Now we subtract 3 from 7 leaving us 4. The answer to the subtraction problem is 46.

$$\begin{array}{r} 7\;{}^{1}2 \\ -{}^{1}2\;\;6 \\ \hline 4\;\;6 \end{array}$$

5. To check our subtraction, we add the subtractor and the remainder, which give us the minuend.

$$\begin{array}{r} 26 \\ 46 \\ \hline 72 \end{array}$$

In expanded form, the equal adding or "borrowing" process can be demonstrated as follows:

1.
$$\begin{array}{r} 72 \\ -\underline{26} \end{array}$$

2.
$$\begin{array}{rr} 70 & 2 \\ 20 & 6 \end{array}$$

3.
$$\begin{array}{rl} 70 & 2 + 10 \\ \underline{10 + 20} & \underline{6 \qquad} \end{array}$$

4.
$$\begin{array}{rr} 70 & 12 \\ \underline{-30} & \underline{-\ 6} \\ 40 & 6 \end{array}$$

5. $40 + 6 = 46$

In teaching the technique of equal additions or "borrowing," it is important to use terms which the child can understand so that he can easily recall what must be done. For this reason many teachers have used the terms "borrow and pay back" to describe the procedure. Children seem to take to it (even though it is technically inaccurate) because it reminds children that the borrowing process requires two actions: adding to the "ones" digit in the minuend and to the "tens" digit in the subtrator. A common error in subtraction is forgetting to add one to the "tens" digit in the subtractor. Thus, when the teacher refers to the operation as "borrowing" and "paying back" the child remembers that it is a two-step operation.

In this regard it is wise to get the child in the habit of performing the "borrowing" and the "paying back" in automatic sequence so that the second addition is not forgotten. Showing the child how the process works in the expanded form will help him retain the equal adding principle in his mind. If the child can understand the idea of subtraction by equal adding as demonstrated in the expanded form, you can refer to the process as "equal adding" rather than "borrowing and paying back." In this instance, an intellectual understanding of the process may be a much better way of helping the child master subtraction than rote memory.

Step 33: Have the child practice the technique of equal adding in these subtractions.

92	64	60	75	88	96
-34	-27	-35	-59	-59	-48

73	52	80	66	77
-46	-18	-67	-37	-38

Step 34: Equal adding in subtractions in which the minuend is a three-digit number and the subtractor a two-digit number. Explain to the pupil that the zero in the hundreds column of the subtractor is invisible.

194	262	460	375	688	496
-67	-56	-49	-38	-69	-38

873	152	380	266	477
-47	-35	-42	-59	-68

Step 35: Subtractions with three-digit numbers without equal adding or "borrowing."

$$\begin{array}{r}684\\-362\\\hline\end{array}\qquad\begin{array}{r}795\\-674\\\hline\end{array}\qquad\begin{array}{r}982\\-871\\\hline\end{array}\qquad\begin{array}{r}588\\-425\\\hline\end{array}\qquad\begin{array}{r}827\\-605\\\hline\end{array}$$

$$\begin{array}{r}757\\-423\\\hline\end{array}\qquad\begin{array}{r}926\\-604\\\hline\end{array}\qquad\begin{array}{r}498\\-320\\\hline\end{array}\qquad\begin{array}{r}629\\-413\\\hline\end{array}$$

Step 36: Subtractions with three-digit numbers with equal adding or "borrowing" in the "ones," "tens," and "hundreds" columns. The subtraction is performed in the following steps:

1. $\begin{array}{r}621\\-267\\\hline\end{array}\qquad\begin{array}{r}6\ \ 2\ \ 1\\-2\ \ 6\ \ 7\\\hline\end{array}$

2. $\begin{array}{r}621\\-267\\\hline\end{array}\qquad\begin{array}{r}6\ \ 2\ {}^{1}1\\-2\ {}^{1}6\ \ 7\\\hline\end{array}$ add 10 here / add 1 here

3. $\begin{array}{r}621\\-267\\\hline 4\end{array}\qquad\begin{array}{r}6\ \ 2\ {}^{1}1\\-2\ {}^{1}6\ \ 7\\\hline 4\end{array}$ subtract 7 from 11

4. $\begin{array}{r}621\\-267\\\hline 4\end{array}\qquad\begin{array}{r}6\ {}^{1}2\ {}^{1}1\\{}^{1}2\ {}^{1}6\ \ 7\\\hline 4\end{array}$ add 10 here / add 1 here

5. $\begin{array}{r}621\\-267\\\hline 54\end{array}\qquad\begin{array}{r}6\ {}^{1}2\ {}^{1}1\\{}^{1}2\ {}^{1}6\ \ 7\\\hline 5\ \ 4\end{array}$ subtract 7 from 12

6. $\begin{array}{r}621\\-267\\\hline 354\end{array}\qquad\begin{array}{r}6\ {}^{1}2\ {}^{1}1\\-{}^{1}2\ {}^{1}6\ \ 7\\\hline 3\ \ 5\ \ 4\end{array}$ subtract 3 from 6

7.
```
  267
  354
  ---
  621     checking the subtraction
```

Step 37: Subtraction exercises with three-digit numbers requiring equal adding. Help the child where he may have difficulty subtracting with zeros.

```
 523    604    830    946    712    657
-298   -387   -266   -377   -475   -289
----   ----   ----   ----   ----   ----

 750    822    905    574    631
-253   -366   -306   -196   -258
----   ----   ----   ----   ----
```

Step 38: Multiplication. Explain to the pupil that multiplication is a short way of adding a lot of the same numbers. For example, in addition we write:

2 + 2 + 2 + 2 + 2 = 10

In multiplication we write the same thing as:

5 X 2 = 10

We use the X to signify multiplication. Here's another example. In addition we write:

4 + 4 + 4 + 4 = 16

In multiplication we write the same thing as:

4 X 4 = 16

Explain that multiplication saves us a lot of time. It is a very useful way of solving many simple problems. Supposing you had six boxes of pencils in which there were six pencils to each box and you wanted to know how many pencils you had altogether. You could add up six sixes in this way:

$$\begin{array}{r} 6 \\ 6 \\ 6 \\ 6 \\ 6 \\ \underline{6} \\ 36 \end{array}$$

Or you could multiply 6 x 6 to get the same answer. The multiplication table gives us the answers to all multiplications with two numbers up to 12. The reason why we go as high as 12 is because 12 is a very common number in our measurement systems. There are 12 months to a year, 12 in a dozen, 12 inches to a foot.

Step 39: The multiplication table. There are a number of ways of learning the multiplication facts. One way to do it is to learn to count by twos, threes, fours, etc. Twos, fives, and tens are the easiest. Once they are learned, start the child learning to count by threes, fours, sixes, sevens, eights, and nines in conjunction with the learning of the multiplication facts. Save elevens and twelves until the pupil has fully mastered the multiplications through nine. Elevens and twelves are not that often used. Thus the child can learn to count by elevens to the ninth place and twelves to the eighth place, which is more than he will ever need in daily use. The multiplication table is an excellent guide to counting by numbers 2 through 12:

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

Step 40: The multiplication facts. The multiplication facts can be taught in equation or column form. For the purpose of verbal memorization, the equation is the better form in that it is easier to "read" rhetorically. Therefore, in teaching the child to memorize the multiplication facts we suggest using such sentences as "two times two is four; two times three is six." Some teachers prefer to say "two twos are four" or "three sixes are eighteen." This is perfectly good form except that it gives a more additive connotation to the process. By using the word "times" the idea of multiplication as distinct from addition is conveyed. Actually, multiplication is merely the memorization of multiple arithmetic facts or counting by numbers over one. The tutor should bear this in mind when teaching a child who is having dif-

ficulty grasping the concept of multiplication—which is really the concept of multiples, or specific quantities counted forward in multiples. Division, on the other hand, deals with specific quantities counted backward in multiples.

The multiplication facts are memorized rhetorically in two ways, either in such sentences as "two times two is four" or in counting as "two, four, six, eight," etc. If both methods are used, the child will have his mastery doubly reinforced. The key to addition is counting by ones. The key to multiplication is counting by numbers over one. Here are the multiplication facts in equation form. Note that we place multiplication by one at the end because some children have trouble understanding what we mean by 1 x 1 = 1 before they've grasped the concept of multiplication.

2 x 1 = 2	3 x 1 = 3	4 x 1 = 4
2 x 2 = 4	3 x 2 = 6	4 x 2 = 8
2 x 3 = 6	3 x 3 = 9	4 x 3 = 12
2 x 4 = 8	3 x 4 = 12	4 x 4 = 16
2 x 5 = 10	3 x 5 = 15	4 x 5 = 20
2 x 6 = 12	3 x 6 = 18	4 x 6 = 24
2 x 7 = 14	3 x 7 = 21	4 x 7 = 28
2 x 8 = 16	3 x 8 = 24	4 x 8 = 32
2 x 9 = 18	3 x 9 = 27	4 x 9 = 36

5 x 1 = 5	6 x 1 = 6	7 x 1 = 7
5 x 2 = 10	6 x 2 = 12	7 x 2 = 14
5 x 3 = 15	6 x 3 = 18	7 x 3 = 21
5 x 4 = 20	6 x 4 = 24	7 x 4 = 28
5 x 5 = 25	6 x 5 = 30	7 x 5 = 35
5 x 6 = 30	6 x 6 = 36	7 x 6 = 42
5 x 7 = 35	6 x 7 = 42	7 x 7 = 49
5 x 8 = 40	6 x 8 = 48	7 x 8 = 56
5 x 9 = 45	6 x 9 = 54	7 x 9 = 63

8 x 1 = 8	9 x 1 = 9	1 x 1 = 1
8 x 2 = 16	9 x 2 = 18	1 x 2 = 2
8 x 3 = 24	9 x 3 = 27	1 x 3 = 3
8 x 4 = 32	9 x 4 = 36	1 x 4 = 4
8 x 5 = 40	9 x 5 = 45	1 x 5 = 5
8 x 6 = 48	9 x 6 = 54	1 x 6 = 6
8 x 7 = 56	9 x 7 = 63	1 x 7 = 7
8 x 8 = 64	9 x 8 = 72	1 x 8 = 8
8 x 9 = 72	9 x 9 = 81	1 x 9 = 9

Note that multiplying by one is really another way of counting by ones. Thus, when we say 1 x 1 = 1, etc., we are simply saying that one one is one, one two is two, one three is three. However, the child must know these multiplication-by-one facts because he will require them in multiplying numbers with more than one digit. Note how the products of multiples of three or six or nine give us the patterns of counting in these multiples. The pupil should be encouraged to memorize these multiple counting patterns as they will be extremely useful in remembering multiplication facts. Also note that any multiplication fact can be demonstrated by writing it out in addition form—in a column or equation. Thus,

$$5 \times 9 = 45$$

or

$$9 + 9 + 9 + 9 + 9 = 45$$

or

$$\begin{array}{r} 9 \\ 9 \\ 9 \\ 9 \\ \underline{9} \\ 45 \end{array}$$

It can also be written out as a position in a counting sequence by nines:

9 18 27 36 45

Step 41: Multiplication by zero. Since zero appears in many numbers of two or more digits, the pupil will have to know how to multiply by zero as follows:

$$
\begin{array}{ll}
0 \times 0 = 0 & 0 \times 0 = 0 \\
0 \times 1 = 0 & 1 \times 0 = 0 \\
0 \times 2 = 0 & 2 \times 0 = 0 \\
0 \times 3 = 0 & 3 \times 0 = 0 \\
0 \times 4 = 0 & 4 \times 0 = 0 \\
0 \times 5 = 0 & 5 \times 0 = 0 \\
0 \times 6 = 0 & 6 \times 0 = 0 \\
0 \times 7 = 0 & 7 \times 0 = 0 \\
0 \times 8 = 0 & 8 \times 0 = 0 \\
0 \times 9 = 0 & 9 \times 0 = 0
\end{array}
$$

You can demonstrate these equations by translating them into addition. Since zero means the absence of quantity, that is, nothing, point out that one nothing equals nothing and nine nothings still equal nothing. You can demonstrate this by writing out an equation of five zeros:

$$0 + 0 + 0 + 0 + 0 = 0$$

Point out that there are five zeros, but they all add up to zero. Point out that 0 x 5 is another way of saying no five, or nothing. In other words when you multiply anything by zero or multiply zeros you still get zero.

Step 42: Familiarize the pupil with the multiplication facts in column form in anticipation of his performing written multiplication problems:

1	2	3	4	5	6	7	8	9
2	2	2	2	2	2	2	2	2
2	4	6	8	10	12	14	16	18
1	2	3	4	5	6	7	8	9
3	3	3	3	3	3	3	3	3
3	6	9	12	15	18	21	24	27
1	2	3	4	5	6	7	8	9
4	4	4	4	4	4	4	4	4
4	8	12	16	20	24	28	32	36
1	2	3	4	5	6	7	8	9
5	5	5	5	5	5	5	5	5
5	10	15	20	25	30	35	40	45
1	2	3	4	5	6	7	8	9
6	6	6	6	6	6	6	6	6
6	12	18	24	30	36	42	48	54
1	2	3	4	5	6	7	8	9
7	7	7	7	7	7	7	7	7
7	14	21	28	35	42	49	56	63
1	2	3	4	5	6	7	8	9
8	8	8	8	8	8	8	8	8
8	16	24	32	40	48	56	64	72
1	2	3	4	5	6	7	8	9
9	9	9	9	9	9	9	9	9
9	18	27	36	45	54	63	72	81

1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9

Note how graphically we see that multiplication is really a form of counting in multiples. Each line of products gives us the counting sequence of the multiplier, starting with one, which is our basic counting sequence by one, the foundation on which the entire structure of arithmetic rests.

Step 43: Multiplication terms.

5	multiplicand
2	multiplier
10	product

Step 44: Multiplication without carrying. Teach the pupil to perform these multiplications which give him practice in the multiplication facts. Point out that the digits in the multiplicand are multiplied from right to left in the same sequence that additions and subtractions are performed. You can give the pupil these multiplications as soon as he has acquired a preliminary mastery of the multiplication facts by verbal drilling and flash cards.

11	21	31	41	51	61	71	81	91	12
2	2	2	2	2	2	2	2	2	2
22	32	42	52	62	72	82	92	13	23
2	2	2	2	2	2	2	2	2	2
33	43	53	63	73	83	93	14	24	34
2	2	2	2	2	2	2	2	2	2
44	54	64	74	84	94	40	50	60	70
2	2	2	2	2	2	2	2	2	2

80	90	21	81	42	91	41	81	63	61
2	2	5	7	4	8	6	9	3	7
51	72	51	72	61	81	51	62	41	83
5	4	8	3	9	6	7	4	5	2
71	61	71	41	81	41	92	71	61	71
8	6	9	7	8	9	2	4	8	7
30	52	60	80	40	52	90	90	30	20
9	4	2	5	8	2	7	3	7	8
82	30	90	90	42	50	51	90	30	70
3	6	5	9	3	9	6	4	5	6
31	32	91	21	20	21	52	61	71	82
8	4	6	9	7	6	3	5	5	4

Step 45: Multiplication with carrying. Since carrying in multiplication is similar to carrying in addition, the pupil should have little difficulty in understanding or catching on to the process. The same place-value principles apply. For example, in adding five fifteens, we do as follows:

			2		2
1.	15	2.	15	3.	15
	15		15		15
	15		15		15
	15		15		15
	15		15		15
			5		75

In multiplying 15 x 5, we do as follows:

$$
\begin{array}{llllll}
 & & & 2 & & 2 \\
1. & 15 & 2. & 15 & 3. & 15 \\
 & \underline{5} & & \underline{5} & & \underline{5} \\
 & & & 5 & & 75
\end{array}
$$

Note that we add the carried 2 to the product of the 5 x 1 in the tens column. Thus, the carrying process actually combines multiplication with addition.

Step 46: Multiplication exercises with carrying. The more difficult multiplication facts are given greater emphasis toward the end.

65	46	57	79	63	32	84	45	35
2	3	5	2	4	6	2	4	3
43	32	22	26	27	25	22	43	26
5	2	8	3	5	7	9	6	4
28	46	47	53	52	96	74	28	53
3	4	2	5	4	2	3	5	8
45	35	49	28	54	35	37	23	96
6	3	5	2	7	9	4	7	3
67	42	89	24	37	29	28	19	62
5	9	4	8	4	7	5	3	8
46	35	27	62	38	62	46	17	37
2	9	3	4	2	9	6	8	5

35	18	49	68	59	84	74	57	87
8	4	5	7	6	3	9	2	6
16	84	49	91	35	19	37	57	59
5	8	4	9	3	2	7	7	4
98	72	69	95	79	84	68	48	97
9	6	3	5	8	8	6	3	9
79	74	86	79	48	69	93	98	34
7	4	7	6	9	8	3	4	7
73	36	43	63	76	43	39	93	67
8	9	6	4	3	3	8	7	6
74	97	98	73	43	48	73	48	76
7	3	6	4	6	8	9	9	8

It is important to become aware of the pupil's working habits as he performs these multiplications. See which ones he performs easily and those that give him trouble. At the heart of any difficulty will probably be a weakness in the more difficult multiplication or addition facts. The remedy is to go back to the facts and drill them more thoroughly. Children pick up poor arithmetic habits as a result of their trying to overcome weaknesses in their knowledge of the basic facts when trying to solve difficult problems. The problem of 44 x 2 is, in essence, actually no more difficult than 99 x 7. The difficulty of the latter lies in the fact that we don't bother to put the same effort in learning to count by nines that we do in learning to count by ones and twos. Thus, we may go through life having trouble multiplying with nines because we were not given enough drill with nines in the primary grades to give us the kind of automatic knowledge which makes arithmetic as effortless as possible. Therefore, we suggest that more time be spent on automatizing arithmetic knowledge than performing problems of no consequence. The problems should be given as tests of knowledge, not as means of reinforcing bad arithmetic habits. Therefore it is much more important to find out how a pupil performs his arithmetic problems than merely to check for correct answers. If a correct answer is arrived at by way of tortuous unit-counting, it should alert us to the remedial work which should be done at this stage of learning.

Step 47: Division. Just as multiplication is a form of counting forward in multiples, division is basically counting backward in multiples. We start with a maximum quantity which we reduce by a divisor to a single smaller multiple. For example, if you want to divide a box of 250 oranges among five people, you divide 250 by 5, giving each fifty. In other words, the 250 has been divided into multiples of fifty. We apply a reverse knowledge of the multiplication facts to achieve the answer. We know that 5 x 5 = 25. In reverse we know that 25 ÷ 5 = 5. Therefore, since our basic division facts are reverses of the multiplication facts, they should be learned together. The pupil's knowledge of his multiplication facts will help him learn the division facts. To help the child understand the practical uses of division, describe problems in which, for various reasons, quantities are divided into multiples. For example, if three children were running a lemonade stand as equal partners and took in 90 cents that day, how would they determine how much each partner had earned? They would use division to find out. When the child has grasped the idea of division, have him learn the division facts.

2 x 1 = 2	2 ÷ 2 = 1	3 x 1 = 3	3 ÷ 3 = 1
2 x 2 = 4	4 ÷ 2 = 2	3 x 2 = 6	6 ÷ 3 = 2
2 x 3 = 6	6 ÷ 2 = 3	3 x 3 = 9	9 ÷ 3 = 3
2 x 4 = 8	8 ÷ 2 = 4	3 x 4 = 12	12 ÷ 3 = 4
2 x 5 = 10	10 ÷ 2 = 5	3 x 5 = 15	15 ÷ 3 = 5
2 x 6 = 12	12 ÷ 2 = 6	3 x 6 = 18	18 ÷ 3 = 6

2 x 7 = 14	14 ÷ 2 = 7	3 x 7 = 21	21 ÷ 3 = 7
2 x 8 = 16	16 ÷ 2 = 8	3 x 8 = 24	24 ÷ 3 = 8
2 x 9 = 18	18 ÷ 2 = 9	3 x 9 = 27	27 ÷ 3 = 9

4 x 1 = 4	4 ÷ 4 = 1	5 x 1 = 5	5 ÷ 5 = 1
4 x 2 = 8	8 ÷ 4 = 2	5 x 2 = 10	10 ÷ 5 = 2
4 x 3 = 12	12 ÷ 4 = 3	5 x 3 = 15	15 ÷ 5 = 3
4 x 4 = 16	16 ÷ 4 = 4	5 x 4 = 20	20 ÷ 5 = 4
4 x 5 = 20	20 ÷ 4 = 5	5 x 5 = 25	25 ÷ 5 = 5
4 x 6 = 24	24 ÷ 4 = 6	5 x 6 = 30	30 ÷ 5 = 6
4 x 7 = 28	28 ÷ 4 = 7	5 x 7 = 35	35 ÷ 5 = 7
4 x 8 = 32	32 ÷ 4 = 8	5 x 8 = 40	40 ÷ 5 = 8
4 x 9 = 36	36 ÷ 4 = 9	5 x 9 = 45	45 ÷ 5 = 9

6 x 1 = 6	6 ÷ 6 = 1	7 x 1 = 7	7 ÷ 7 = 1
6 x 2 = 12	12 ÷ 6 = 2	7 x 2 = 14	14 ÷ 7 = 2
6 x 3 = 18	18 ÷ 6 = 3	7 x 3 = 21	21 ÷ 7 = 3
6 x 4 = 24	24 ÷ 6 = 4	7 x 4 = 28	28 ÷ 7 = 4
6 x 5 = 30	30 ÷ 6 = 5	7 x 5 = 35	35 ÷ 7 = 5
6 x 6 = 36	36 ÷ 6 = 6	7 x 6 = 42	42 ÷ 7 = 6
6 x 7 = 42	42 ÷ 6 = 7	7 x 7 = 49	49 ÷ 7 = 7

6 x 8 = 48	48 ÷ 6 = 8	7 x 8 = 56	56 ÷ 7 = 8
6 x 9 = 54	54 ÷ 6 = 9	7 x 9 = 63	63 ÷ 7 = 9

8 x 1 = 8	8 ÷ 8 = 1	9 x 1 = 9	9 ÷ 9 = 1
8 x 2 = 16	16 ÷ 8 = 2	9 x 2 = 18	18 ÷ 9 = 2
8 x 3 = 24	24 ÷ 8 = 3	9 x 3 = 27	27 ÷ 9 = 3
8 x 4 = 32	32 ÷ 8 = 4	9 x 4 = 36	36 ÷ 9 = 4
8 x 5 = 40	40 ÷ 8 = 5	9 x 5 = 45	45 ÷ 9 = 5
8 x 6 = 48	48 ÷ 8 = 6	9 x 6 = 54	54 ÷ 9 = 6
8 x 7 = 56	56 ÷ 8 = 7	9 x 7 = 63	63 ÷ 9 = 7
8 x 8 = 64	64 ÷ 8 = 8	9 x 8 = 72	72 ÷ 9 = 8
8 x 9 = 72	72 ÷ 8 = 9	9 x 9 = 81	81 ÷ 9 = 9

Step 48: Division terms. The most important fact in any division problem is the quantity or number you start with—the *dividend*, or that which is to be divided. The Latin word *dividere*, from which we derive our term division, means to separate, divide, distribute. In arithmetic it means to separate into equal parts by a *divisor*. Thus, in division we start with a quantity which is to be divided or distributed into equal parts or multiples by a *divisor*. Since not all numbers or quantities can be divided into equal parts, we may have a few units left over. For example, 252 oranges divided among 5 people gives us 50 oranges each with 2 left over. In division we call the 50 the *quotient* and the 2 left over the *remainder*. This is not to be confused with the remainder in subtraction, which is also called the difference in order to avoid confusing it with the remainder in division.

Step 49: Division skills should be developed through sets of exercises which become progressively more complex. We start with a review of the primary division facts, proceeding into exercises with one-digit divisors but two- and three-digit quotients without carrying.

$4\overline{)32}$	$2\overline{)18}$	$3\overline{)27}$	$4\overline{)24}$	$5\overline{)35}$	$7\overline{)49}$	$8\overline{)56}$
$6\overline{)48}$	$9\overline{)72}$	$8\overline{)64}$	$9\overline{)63}$	$8\overline{)72}$	$4\overline{)88}$	$3\overline{)69}$
$7\overline{)357}$	$6\overline{)486}$	$9\overline{)549}$	$3\overline{)216}$	$8\overline{)408}$	$2\overline{)148}$	$4\overline{)248}$
$5\overline{)205}$	$7\overline{)427}$	$5\overline{)255}$	$3\overline{)189}$	$9\overline{)729}$	$6\overline{)246}$	$8\overline{)728}$
$4\overline{)168}$	$7\overline{)567}$	$5\overline{)105}$	$3\overline{)156}$	$5\overline{)305}$	$6\overline{)126}$	$7\overline{)147}$
$9\overline{)189}$	$6\overline{)546}$	$4\overline{)128}$	$8\overline{)248}$	$6\overline{)426}$	$5\overline{)155}$	$4\overline{)364}$
$6\overline{)306}$	$9\overline{)459}$	$3\overline{)126}$	$9\overline{)819}$	$5\overline{)455}$	$6\overline{)186}$	$3\overline{)246}$
$8\overline{)168}$	$7\overline{)217}$	$3\overline{)273}$	$7\overline{)637}$	$2\overline{)104}$	$8\overline{)328}$	$5\overline{)405}$
$2\overline{)122}$	$4\overline{)208}$	$9\overline{)279}$	$7\overline{)497}$	$8\overline{)488}$	$4\overline{)284}$	$2\overline{)184}$
$9\overline{)369}$	$8\overline{)648}$	$7\overline{)287}$	$9\overline{)639}$	$8\overline{)568}$	$4\overline{)328}$	$5\overline{)355}$

Step 50: Primary division facts with remainders. Explain to the pupil that it is not always possible to divide a number evenly. Go back to the lemonade stand example and ask how much would each partner have gotten if they had earned only 29 cents instead of 90. We divide 29 by 3 to find out:

1. $3\overline{)29}$

2. To divide 29 by 3 we look for the number closest to 29, but not higher than 29 that can be divided by 3—or that 3 "goes into." That number is 27. Thirty obviously is too high. So we write 9 as the quotient, multiply it by the divisor—3. This gives us the dividend closest to 29 which can be evenly divided into three multiples. We write that dividend under the original dividend of 29, thus:

$$\begin{array}{r} 9 \\ 3\overline{)29} \\ 27 \end{array}$$

3. Then we subtract 27 from 29, which gives us the remainder, or how much extra is left over:

$$\begin{array}{r} 9 \\ 3\overline{)29} \\ \underline{27} \\ 2 \end{array}$$

To check division we multiply the quotient by the divisor and add the remainder. This gives us the dividend.

What do the partners do with the extra 2 cents? Perhaps some candy shop might sell them something which they can then divide into three parts.

Here are a series of division exercises with remainders. Some children will have difficulty finding the right divisible number into which the divisor can go. Let the child refer to the division table for help if necessary. If he has learned to count by numbers over two, this knowledge will be useful in finding the closest divisible number. Have the pupil write out the full process and check each division answer.

$2\overline{)5}$	$2\overline{)7}$	$2\overline{)9}$	$2\overline{)11}$	$2\overline{)15}$	$2\overline{)17}$	$2\overline{)19}$
$3\overline{)7}$	$3\overline{)10}$	$3\overline{)13}$	$3\overline{)19}$	$3\overline{)22}$	$3\overline{)25}$	$3\overline{)29}$
$4\overline{)6}$	$4\overline{)9}$	$4\overline{)15}$	$4\overline{)21}$	$4\overline{)27}$	$4\overline{)30}$	$4\overline{)39}$
$5\overline{)13}$	$5\overline{)22}$	$5\overline{)31}$	$5\overline{)37}$	$5\overline{)39}$	$5\overline{)43}$	$5\overline{)48}$
$6\overline{)15}$	$6\overline{)19}$	$6\overline{)21}$	$6\overline{)28}$	$6\overline{)33}$	$6\overline{)37}$	$6\overline{)40}$
$6\overline{)45}$	$6\overline{)50}$	$6\overline{)55}$	$6\overline{)57}$	$6\overline{)59}$		
$7\overline{)13}$	$7\overline{)16}$	$7\overline{)20}$	$7\overline{)25}$	$7\overline{)30}$	$7\overline{)34}$	$7\overline{)40}$
$7\overline{)47}$	$7\overline{)51}$	$7\overline{)55}$	$7\overline{)58}$	$7\overline{)62}$	$7\overline{)65}$	$7\overline{)68}$
$8\overline{)18}$	$8\overline{)28}$	$8\overline{)31}$	$8\overline{)44}$	$8\overline{)47}$	$8\overline{)51}$	$8\overline{)55}$
$8\overline{)58}$	$8\overline{)60}$	$8\overline{)65}$	$8\overline{)68}$	$8\overline{)70}$	$8\overline{)73}$	$8\overline{)77}$
$9\overline{)12}$	$9\overline{)17}$	$9\overline{)21}$	$9\overline{)26}$	$9\overline{)32}$	$9\overline{)35}$	$9\overline{)40}$
$9\overline{)44}$	$9\overline{)48}$	$9\overline{)52}$	$9\overline{)55}$	$9\overline{)60}$	$9\overline{)62}$	$9\overline{)65}$
$9\overline{)69}$	$9\overline{)71}$	$9\overline{)75}$	$9\overline{)79}$	$9\overline{)83}$	$9\overline{)85}$	$9\overline{)87}$

Step 51: Division with carrying, with and without remainders. The carrying process is performed by the process of "bringing down" as demonstrated in the examples below. An X is placed under the number brought down to indicate the bringing down process.

1. Without a remainder

$$3\overline{)84}$$

2. How many 3's go into 8? Two.

$$\begin{array}{r} 2\\ 3\overline{)84}\\ \underline{6}\\ 2\end{array}$$

3. Bring down 4

$$\begin{array}{r} 2\\ 3\overline{)84}\\ \underline{6^{\times}}\\ 24\end{array}$$

4. How many 3's go into 24? Eight.

$$\begin{array}{r} 28\\ 3\overline{)84}\\ \underline{6^{\times}}\\ 24\\ \underline{24}\end{array}$$

1. With a remainder

$$3\overline{)89}$$

2.

$$\begin{array}{r} 2\\ 3\overline{)89}\\ \underline{6}\\ 2\end{array}$$

3. Bring down 9

$$\begin{array}{r} 2\\ 3\overline{)89}\\ \underline{6^{\times}}\\ 29\end{array}$$

4. How many 3's go into 29? Nine, with a remainder of 2.

$$\begin{array}{r} 29\\ 3\overline{)89}\\ \underline{6^{\times}}\\ 29\\ \underline{27}\\ 2\end{array}$$

The following exercises give practice in division facts with carrying, but without remainders:

$2\overline{)56}$	$2\overline{)74}$	$2\overline{)138}$	$2\overline{)156}$	$2\overline{)178}$	$2\overline{)194}$
$3\overline{)78}$	$3\overline{)81}$	$3\overline{)195}$	$3\overline{)234}$	$3\overline{)258}$	$3\overline{)297}$
$4\overline{)76}$	$4\overline{)96}$	$4\overline{)268}$	$4\overline{)296}$	$4\overline{)356}$	$4\overline{)392}$
$5\overline{)95}$	$5\overline{)135}$	$5\overline{)190}$	$5\overline{)245}$	$5\overline{)315}$	$5\overline{)420}$
$6\overline{)156}$	$6\overline{)198}$	$6\overline{)264}$	$6\overline{)396}$	$6\overline{)444}$	$6\overline{)594}$
$7\overline{)196}$	$7\overline{)252}$	$7\overline{)315}$	$7\overline{)406}$	$7\overline{)553}$	$7\overline{)679}$
$8\overline{)256}$	$8\overline{)352}$	$8\overline{)472}$	$8\overline{)544}$	$8\overline{)696}$	$8\overline{)784}$
$9\overline{)351}$	$9\overline{)477}$	$9\overline{)585}$	$9\overline{)702}$	$9\overline{)774}$	$9\overline{)891}$

The following exercises give practice in division with carrying and remainders. Make sure that the pupil places each figure of the quotient directly over the last figure of the partial dividend being used. A way for the pupil to keep track of the dividend digits brought down is to place a small x mark under the dividend digit as it is brought down.

$3\overline{)206}$	$7\overline{)325}$	$5\overline{)172}$	$2\overline{)31}$	$6\overline{)224}$	$4\overline{)311}$
$8\overline{)217}$	$4\overline{)139}$	$9\overline{)716}$	$3\overline{)74}$	$6\overline{)334}$	$7\overline{)132}$
$9\overline{)318}$	$5\overline{)284}$	$6\overline{)115}$	$2\overline{)75}$	$3\overline{)239}$	$8\overline{)412}$

$7\overline{)197}$	$3\overline{)172}$	$9\overline{)832}$	$4\overline{)50}$	$5\overline{)391}$	$6\overline{)279}$
$4\overline{)387}$	$5\overline{)473}$	$8\overline{)306}$	$2\overline{)97}$	$7\overline{)389}$	$3\overline{)40}$
$7\overline{)261}$	$9\overline{)415}$	$8\overline{)395}$	$5\overline{)63}$	$6\overline{)169}$	$8\overline{)129}$
$2\overline{)119}$	$4\overline{)225}$	$9\overline{)732}$	$2\overline{)53}$	$8\overline{)765}$	$5\overline{)116}$

Step 52: Three-digit quotients. The following practice examples require two carrying operations. Some come out even, but most have remainders. Make sure the pupil places the quotient figures in their proper position over the dividend and places a small **x** mark under each digit in the dividend as it is brought down. This will help him keep track of each step of the operation.

$5\overline{)933}$	$3\overline{)2810}$	$2\overline{)972}$	$6\overline{)5591}$	$8\overline{)3087}$	$4\overline{)1077}$	$9\overline{)2550}$
$2\overline{)719}$	$7\overline{)4490}$	$3\overline{)741}$	$9\overline{)6831}$	$6\overline{)1724}$	$8\overline{)1395}$	$6\overline{)2740}$
$4\overline{)734}$	$4\overline{)2299}$	$3\overline{)556}$	$7\overline{)1986}$	$5\overline{)2862}$	$7\overline{)6705}$	$9\overline{)5776}$
$2\overline{)592}$	$5\overline{)1974}$	$4\overline{)931}$	$8\overline{)2156}$	$7\overline{)4545}$	$9\overline{)7097}$	$6\overline{)5921}$

Step 53: Zeros in the quotient. Some children find it difficult to deal with zeros in the quotient. The following exercises are designed to give the child practice with handling zeros in the quotient. The rule to remember is that every time a digit in the dividend is brought down, a figure must be written in the quotient.

$2\overline{)40}$	$2\overline{)204}$	$2\overline{)2004}$	$2\overline{)41}$	$3\overline{)62}$	$2\overline{)410}$
$3\overline{)621}$	$3\overline{)322}$	$3\overline{)622}$	$4\overline{)819}$	$4\overline{)962}$	$5\overline{)654}$
$6\overline{)605}$	$7\overline{)4214}$	$8\overline{)7265}$	$9\overline{)8156}$	$7\overline{)3563}$	$6\overline{)4838}$
$5\overline{)1043}$	$8\overline{)5610}$	$9\overline{)5436}$	$3\overline{)3232}$	$4\overline{)3625}$	$6\overline{)4082}$
$7\overline{)286}$	$8\overline{)3841}$	$5\overline{)901}$	$9\overline{)187}$	$7\overline{)3546}$	$9\overline{)6316}$

Note this fundamental principle: zero divided by any number equals zero. Also, no number can be divided by zero for the simple reason that something cannot be divided by nothing.

Step 54: Fractions. Children start understanding concepts of fractions quite early, as words and expressions like "a half" and "a quarter" become part of their vocabulary. But they also speak in terms of the "bigger half" and the "smaller half," which means that their understanding of fractions as a precise arithmetic concept is still lacking. An arithmetic fraction, in symbolic terms, represents one or more of the equal parts of a whole. The tutor should find out how much the child does know about fractions and then help the child to understand them in specific arithmetic terms. Since operations with fractions are not taught until the fourth grade, this primer will merely suggest a sequence of instruction leading into operations with fractions. The purpose of the instruction here is to find out how much the child already knows about fractions and to expand on that.

Since decimals are merely another way of writing fractions in place-value notation, you might teach some elementary facts about decimals, particularly in relation to money. (See Step 61.) The pupil will no doubt know by now that there are 100 cents in a dollar and that fifty cents are a half dollar and twenty-five cents are a quarter.

First, it is necessary to make sure that the child grasps the concept of a fraction—that is, a part of a whole or an aggregate. One pie can be divided into two, three, four parts, etc. A dozen eggs can be divided into a half dozen. One dollar can be divided into two halves, four quarters, ten dimes, twenty nickels, one hundred cents. Since the use of money is already in the child's experience, you might teach the fractions 1/2, 1/4, and 1/10 in relation to money. Fractions such as 1/3, 1/8, 1/5, etc., can be taught in relation to dividing up a whole pie into a number of pieces.

Fractions should also be taught in relation to line measurement: 1/2 inch, 1/4 inch, etc.; and in relation to weight: 1/2 pound, 1/4 pound; in relation to liquid measurement: 1/2 gallon, 1/2 pint; and in relation to time: half-hour, quarter-hour.

The first step is to teach the child to read fractions. Start with:

1/2 1/3 1/4 1/5 1/6 1/7 1/8 1/9 1/10

Demonstrate these fractions on the blackboard by dividing a pie or a candy bar into fractional parts. A fraction is one or more of the equal parts of a whole. We call the number above or to the left of the stroke the numerator and the number below or to the right of the stroke the denominator. The denominator tells us how many parts the whole has been divided into. It also tells us the comparative size of the parts—if we are dealing with wholes of similar size. The numerator tells us how many parts have been taken. A unit fraction, such as those above, is a fraction whose numerator is 1. The child should note from the above fractions that the larger the demoninator, the smaller the fraction. The larger the numerator, the larger the fraction. You can use blackboard drawings to demonstrate this.

It is important for the child to understand the difference between the numerator and the denominator and their relation to each other. This can be done by demonstrating that the numerator and the denominator of a fraction can be multiplied or divided by the same number without changing the value of the fraction. For example, demonstrate how 1/2 is the same as 2/4 or 5/10. But explain that we use fractions in their lowest terms by reducing 2/4 and 5/10 to 1/2. You can demonstrate this on the blackboard by dividing up a rectangle into two halves, four fourths, ten tenths, and showing how one half is the equivalent of two fourths or five tenths.

Step 55: Adding unit fractions. By teaching the child to add unit fractions we can acquaint him with other common fractions. We add fractions with the same denominators by adding the numerators only. For example:

1/3 + 1/3 = 2/3

1/4 + 1/4 = 2/4 = 1/2

Notice how in adding 1/4 + 1/4 we reduced the 2/4 to its lowest terms, 1/2.

1/4 + 1/4 + 1/4 = 3/4

1/8 + 1/8 + 1/8 = 3/8

1/5 + 1/5 = 2/5

1/5 + 1/5 + 1/5 = 3/5

1/5 + 1/5 + 1/5 + 1/5 = 4/5

1/8 + 1/8 + 1/8 + 1/8 + 1/8 = 5/8

1/8 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8 = 7/8

Adding common fractions. Show how some of the above fractions can be obtained as follows:

2/5 + 1/5 = 3/5

3/5 + 1/5 = 4/5

2/5 + 2/5 = 4/5

3/8 + 3/8 + 1/8 = 7/8

Also demonstrate the following deduced from the above, and see if the pupil can figure out what must be done to add fractions with different denominators:

1/2 + 1/4 = 3/4

1/4 + 1/8 = 3/8

3/8 + 1/4 = 5/8

1/2 + 3/8 = 7/8

Comparing fractions. Have the child compare the size of unit fractions by drawing circles on the blackboard divided into smaller and smaller unit fractions. Then have the child compare the size of 1/2 to 2/3 to 3/4, etc. These exercises will help the child learn that the larger the denominator the smaller the fraction and the larger the numerator the larger the fraction if the denominator remains the same. These exercises also teach the child to understand that when the numerator and the denominator are equal, the fraction always equals 1. For example:

1/2 + 1/2 = 2/2 = 1

1/3 + 1/3 + 1/3 = 3/3 = 1

1/4 + 1/4 + 1/4 + 1/4 = 4/4 = 1

1/5 + 1/5 + 1/5 + 1/5 + 1/5 = 5/5 = 1

Note that fractions in which the numerator exceeds the denominator equal more than one. For example: 5/4 = 1 1/4; 8/7 = 1 1/7; 9/5 = 1 4/5. A whole number (or integer) with a fraction is called a *mixed number.* A fraction in which the numerator is less than the denominator is called a *proper fraction.* A fraction in which the numerator is equal to or greater than the denominator is called an *improper fraction.*

Changing the denominator of a fraction. Comparing fractions also enables the child to discover the equivalence of fractions which have different numerators and different denominators. The child learns how common denominators are found for adding fractions with different denominators, and also how fractions are reduced to their lowest terms. For example, you can show how 6 sixths can be written with different fractions as follows:

$$\underbrace{1/6 + 1/6}_{2/6} + \underbrace{1/6}_{1/6} + \underbrace{1/6 + 1/6 + 1/6}_{3/6} = 6/6 = 1$$

$$1/3 \quad + \quad 1/6 \quad + \quad 1/2 \quad = \quad 6/6 = 1$$

Reducing fractions to their lowest terms. The important principle for the pupil to learn is that the numerator and the denominator of a fraction can be divided by the same number without changing the value of the fraction. In preparing exercises, use those fractions which are obtained as the result of adding or subtracting fractions. Fourths, sixths, eighths, tenths, twelfths, sixteenths, twentieths, and twenty-fourths will occur fairly often. Here are some practice examples to be reduced to their lowest terms:

4/10	5/20	8/12	14/24	4/24	12/16
4/6	2/24	10/20	6/8	8/24	4/12
4/16	3/6	8/16	10/24	2/12	4/8
14/16	18/24	16/24	9/24	6/20	4/20
12/24	16/20	6/24	2/16	8/20	14/20
10/12	15/20	4/16	10/16	6/10	5/10

Here are additional practice examples of fractions to be reduced to their lowest terms. These fractions are not as frequently encountered as those in the first group. In reducing these fractions to lowest terms the pupil can divide the numerator and denominator by two or more numbers. However, to save time the pupil should learn to divide by the largest number that he can see that will divide both the numerator and denominator:

6/15	15/48	30/48	12/15	12/30	9/48	8/64
6/48	24/32	3/18	14/64	5/30	10/25	3/9
4/18	20/32	16/48	15/18	6/9	42/48	4/14
12/18	21/30	24/64	9/15	20/25	10/18	10/15
36/64	2/14	5/25	48/64	16/32	12/32	12/14
26/30	30/32	56/64	15/25	24/30	28/32	10/14
6/18	24/48	20/64	5/15	6/14	10/30	6/32

With the completion of the above instruction the pupil should be ready to start learning to add, subtract, multiply and divide with fractions and mixed numbers. However, since these operations are beyond the scope of the primary grades, the instruction on the pages which follow has been added basically for reference purposes.

Step 56: Adding fractions with different denominators. In order to add fractions with different denominators, we must first make them have the same denominator—or a common denominator. To find the common denominator, we multiply the two denominators with each other, and to get the correct new numerators, we cross multiply the denominators with the numerators. For example:

$$\frac{1}{4} + \frac{1}{3} = \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$$

First we cross multiply 3 x 1 and 4 x 1 in order to get the correct numerators. Then we multiply the two denominators 4 x 3 to get the common denominator. Here are more examples:

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

$$\frac{2}{3} + \frac{4}{5} = \frac{10}{15} + \frac{12}{15} = \frac{22}{15} = 1\frac{7}{15}$$

$$\frac{2}{7} + \frac{3}{8} = \frac{16}{56} + \frac{21}{56} = \frac{37}{56}$$

Step 57: Subtracting fractions. In subtracting fractions we go through the same process of finding the common denominator and the new numerators, and then we merely subtract numerators instead of adding them:

$$\frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} = \frac{1}{12}$$

$$\frac{3}{4} - \frac{2}{5} = \frac{15}{20} - \frac{8}{20} = \frac{7}{20}$$

The following fraction table reveals how the pattern of numerators and denominators is directly related to our multiplication facts. Since the greatest aids to memory in arithmetic are the number patterns, the table is worth studying. It will reinforce the pupil's understanding of the advantages of memorizing counting by numbers over one and knowing the basic multiplication facts cold.

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12} = \frac{7}{14} = \frac{8}{16} = \frac{9}{18} = \frac{10}{20}$$

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15} = \frac{12}{18} = \frac{14}{21} = \frac{16}{24} = \frac{18}{27} = \frac{20}{30}$$

$$\frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{12}{16} = \frac{15}{20} = \frac{18}{24} = \frac{21}{28} = \frac{24}{32} = \frac{27}{36} = \frac{30}{40}$$

$$\frac{4}{5} = \frac{8}{10} = \frac{12}{15} = \frac{16}{20} = \frac{20}{25} = \frac{24}{30} = \frac{28}{35} = \frac{32}{40} = \frac{36}{45} = \frac{40}{50}$$

$$\frac{5}{6} = \frac{10}{12} = \frac{15}{18} = \frac{20}{24} = \frac{25}{30} = \frac{30}{36} = \frac{35}{42} = \frac{40}{48} = \frac{45}{54} = \frac{50}{60}$$

$$\frac{6}{7} = \frac{12}{14} = \frac{18}{21} = \frac{24}{28} = \frac{30}{35} = \frac{36}{42} = \frac{42}{49} = \frac{48}{56} = \frac{54}{63} = \frac{60}{70}$$

$$\frac{7}{8} = \frac{14}{16} = \frac{21}{24} = \frac{28}{32} = \frac{35}{40} = \frac{42}{48} = \frac{49}{56} = \frac{56}{64} = \frac{63}{72} = \frac{70}{80}$$

$$\frac{8}{9} = \frac{16}{18} = \frac{24}{27} = \frac{32}{36} = \frac{40}{45} = \frac{48}{54} = \frac{56}{63} = \frac{64}{72} = \frac{72}{81} = \frac{80}{90}$$

$$\frac{9}{10} = \frac{18}{20} = \frac{27}{30} = \frac{36}{40} = \frac{45}{50} = \frac{54}{60} = \frac{63}{70} = \frac{72}{80} = \frac{81}{90} = \frac{90}{100}$$

Step 58: Adding more than two fractions with different denominators.

1. $\frac{3}{4} + \frac{2}{3} + \frac{5}{6} =$

2. Add the first two fractions:

$$\frac{3}{4} + \frac{2}{3} = \frac{9}{12} + \frac{8}{12} = \frac{17}{12}$$

3. $\frac{17}{12} + \frac{5}{6} = \frac{17}{12} + \frac{10}{12} = \frac{27}{12} = 2\frac{3}{12} = 2\frac{1}{4}$

4. $\frac{3}{4} + \frac{2}{3} + \frac{5}{6} = 2\frac{1}{4}$

Another example:

1. $$\frac{6}{7} + \frac{2}{3} + \frac{4}{5} =$$

2. $$\frac{6}{7} + \frac{2}{3} = \frac{18}{21} + \frac{14}{21} = \frac{32}{21}$$

3. $$\frac{32}{21} + \frac{4}{5} = \frac{160}{105} + \frac{84}{105} = \frac{244}{105} = 2\frac{34}{105} = 2\frac{1}{3}$$

Note that 34/105 is close enough to be considered 1/3.

4. $$\frac{6}{7} + \frac{2}{3} + \frac{4}{5} = 2\frac{1}{3}$$

Step 59: Multiplying with fractions. In multiplying fractions, the numerators are multiplied with one another and the denominators are multiplied with one another. Note that we convert whole numbers into fractions to simplify the process:

$$\frac{1}{2} \times 4 = \frac{1}{2} \times \frac{4}{1} = \frac{4}{2} = 2$$

$$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

$$\frac{3}{4} \times \frac{7}{8} = \frac{21}{32}$$

$$\frac{5}{6} \times 6 = \frac{5}{6} \times \frac{6}{1} = \frac{30}{6} = 5$$

$$\frac{1}{22} \times 11 = \frac{1}{22} \times \frac{11}{1} = \frac{11}{22} = \frac{1}{2}$$

$$\frac{1}{2} \times 1\frac{1}{2} = \frac{1}{2} \times \left[\frac{2}{2} + \frac{1}{2}\right] = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$$

$$2\frac{1}{4} \times 1\frac{5}{8} = \left[\frac{8}{4} + \frac{1}{4}\right] \times \left[\frac{8}{8} + \frac{5}{8}\right] =$$

$$\frac{9}{4} \times \frac{13}{8} = \frac{117}{32} = 3\frac{21}{32}$$

Step 60: Dividing by fractions. Dividing by a fraction is the same as multiplying by the same fraction turned upside down. The upside down fraction is known as the reciprocal. Study the following examples.

$$\frac{1}{2} \div 2 = \frac{1}{2} \div \frac{2}{1} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$\frac{1}{2} \div \frac{1}{2} = \frac{1}{2} \times \frac{2}{1} = \frac{2}{2} = 1$$

$$\frac{3}{4} \div 3 = \frac{3}{4} \times \frac{1}{3} = \frac{3}{12} = \frac{1}{4}$$

$$\frac{3}{4} \div \frac{1}{8} = \frac{3}{4} \times \frac{8}{1} = \frac{24}{4} = 6$$

$$1\frac{1}{2} \div 3\frac{1}{4} = \frac{3}{2} \div \frac{13}{4} = \frac{3}{2} \times \frac{4}{13} = \frac{12}{26} = \frac{6}{13}$$

Checking above division:

$$\frac{6}{13} \times 3\frac{1}{4} = \frac{6}{13} \times \frac{13}{4} = \frac{78}{52} = 1\frac{26}{52} = 1\frac{1}{2}$$

Step 61: Decimals. The simplest way to introduce the pupil to the concept of decimals (that is, fractions expressed in place-value notation) is through an understanding of how we deal with money in decimals. A decimal is a numerator of one or more digits with an unwritten denominator of ten or some power of ten depending on the position of the digit or digits in relation to the decimal point. For example:

.5	=	5/10	.85	=	85/100
.05	=	5/100	.805	=	805/1000
.005	=	5/1000			

In other words, the digit columns to the right of the decimal point represent 10ths, 100ths, 1000ths, etc., in the same order that the digit columns to the left of the decimal point represent "ones," "tens," "hundreds," "thousands," etc. Thus, we can write 555 as:

$$500 + 50 + 5 = 555$$

and we can write .555 as:

$$5/10 + 5/100 + 5/1000 = 500/1000 + 50/1000 + 5/1000 = 555/1000$$

In money notation, where the unwritten denominator is 100, the decimal is only written out to two places. Note that .50 is the same as .5. However, it is written as .50 because we refer to it as 50 cents as well as a half-dollar, and also to avoid confusing it with .05. In general, our decimal money notations are read more as hieroglyphics than place-value notations because of how they are referred to verbally and because of our coin system. Thus:

100¢	$1.00	one dollar or dollar
50¢	.50	half dollar, fifty cents

25¢	.25	quarter, twenty-five cents
10¢	.10	dime, ten cents
5¢	.05	nickel, five cents
1¢	.01	cent

Demonstrate how decimals can be converted into decimal fractions, then reduced to proper fractions:

$$.50 = \frac{50}{100} = \frac{1}{2} \qquad .05 = \frac{5}{100} = \frac{1}{20}$$

$$.25 = \frac{25}{100} = \frac{1}{4} \qquad .01 = \frac{1}{100}$$

$$.10 = \frac{10}{100} = \frac{1}{10}$$

Converting fractions to decimals. To convert or change a common fraction to a decimal is a very simple and straightforward procedure. It consists simply of dividing the numerator (with zeros annexed) by the denominator and placing a decimal point before the proper figure of the quotient. For example:

```
1/2        .5          1/2 = .5
         2)1.0
           1 0
           ---

3/4        .75         3/4 = .75
         4)3.00
           2 8
           ---
            20
            20
            --

1/4        .25         1/4 = .25
         4)1.00
            8
           ---
            20
            20
            --
```

Have the pupil read the following amounts:

$.38	$.09	$.65	$1.25	$1.98
$9.65	$12.25	$25.25	$42.01	$99.99
$175.15	$305.02	$548.23	$690.50	$952.33
$1,000.00	$1,025.08	$1,200.50	$5,493.26	$6,004.10
$4,444.44	$9,500.00	$7,320.01	$10,000.00	$13,679.84

Provide exercises in adding, subtracting, multiplying and dividing monetary sums. You can use a shopping list for additions, regular prices vs. sale prices for subtractions, multiple buying for multiplication (if one cost 3.98, how much will 3 cost?) and unit pricing for division (if the price of the item is 3 for $1.00 or 5 for .88, how much will one cost?).

Teach the pupil to read large sums of money in round figures up to one million:

$100	one hundred
$1,000	one thousand
$10,000	ten thousand
$100,000	one hundred thousand
$1,000,000	one million (one thousand thousand)

Step 62: Linear measurement. Explain the following measurements:

1 inch		
12 inches	=	1 foot
3 feet	=	1 yard
5,280 feet	=	1 mile
1,760 yards	=	1 mile

Teach the child to read the following measurements:

1 foot 2 inches	=	1 ft. 2 in.	= 1' 2"
5 yards	=	5 yds.	
20 miles	=	20 mi.	

Teach the pupil to add, subtract, multiply, and divide feet and inches. Note that in subtraction and division we can simplify computation by converting feet into inches by multiplying feet by 12, subtracting or dividing in inches, then converting back into feet and inches for the final answer.

Addition:

$$\begin{array}{rr} 3' & 10'' \\ 4' & 9'' \\ \hline 7' & 19'' \end{array} \quad = 8'\ 7''$$

Subtraction:

$$\begin{array}{rr} 10' & 8'' \\ -\ 2' & 4'' \\ \hline 8' & 4'' \end{array}$$

Subtraction:

$$\begin{array}{rr} 10' & 2'' \\ -\ 3' & 9'' \\ \hline & \end{array} \quad = \quad \begin{array}{r} 122'' \\ -\ 45'' \\ \hline 77'' \end{array} \quad = \quad \begin{array}{rr} 10' & 2'' \\ -3' & 9'' \\ \hline 6' & 5'' \end{array}$$

Multiplication:

$$\begin{array}{rr} 5' & 3'' \\ \times & 10 \\ \hline 50' & 30'' \end{array} \quad = \quad 52'\ 6''$$

Division:

$$10'\ 8'' \div 4 \quad = \quad \begin{array}{r} 32 \\ 4\overline{)128} \end{array} \quad = 2'\ 8''$$

Computing with fractions of inches can become complicated, especially in division, where conversion to decimals is usually necessary. If the pupil is up to it, show him how it is done with the following examples:

Addition:

$$\begin{array}{rrr} 6' & 5 & 1/2'' \\ 4' & 10 & 3/4'' \\ \hline 10' & 15 & 5/4'' \end{array} \quad = 10'\ 16\ 1/4'' \quad =$$

$$11'\ 4\ 1/4''$$

Subtraction:

$$\begin{array}{rr} 6' & 5\ 3/8'' \\ -2' & 4\ 1/4'' \\ \hline 4' & 1\ 1/8'' \end{array}$$

Subtraction:

$$\begin{array}{rr} 6' & 2\ 3/16'' \\ -3' & 9\ 1/4'' \\ \hline \end{array} = \begin{array}{r} 74\ 3/16'' \\ -45\ 1/4'' \\ \hline \end{array} = \begin{array}{r} 74\ 3/16'' \\ -45\ 4/16'' \\ \hline \end{array} =$$

$$\begin{array}{rr} 73 & 19/16'' \\ -45 & 4/16'' \\ \hline 28 & 15/16'' \end{array} = \begin{array}{rr} 6' & 2\ 3/16'' \\ -3' & 9\ 1/4'' \\ \hline 2' & 4\ 15/16'' \end{array}$$

Multiplication:

$$\begin{array}{rr} 12' & 8\ 3/4'' \\ \times & 18 \\ \hline \end{array} = 216'\ 144''\ 54/4'' =$$

$$216'\ 144''\ 13\ 1/2'' = 216'\ 157\ 1/2'' = 229'\ 1\ 1/2''$$

Division:

$$38'\ 1\ 1/2'' \div 3 = 457\ 1/2'' \div 3 =$$

$$\begin{array}{r} 152.5 \\ 3\overline{)457.5} \end{array} = 12'\ 8\ 1/2''$$

Step 63: Liquid measurement. Undoubtedly the pupil will be familiar with our liquid measurement terms. However, he may not know how many pints make a quart, etc. Devise problems in which pints are converted into quarts and quarts into gallons, etc.

pint		
2 pints	=	1 quart
4 quarts	=	1 gallon

Step 64: Weight. As with liquid measurement, the pupil will be familiar with weight measurement terms. However, teach him the relationship of ounces to pounds and pounds to tons.

ounce		
4 ounces	=	quarter-pound
8 ounces	=	half-pound
16 ounces	=	1 pound
2,000 pounds	=	1 ton

Step 65: Time. Find out how much the child knows about telling time. Then explain our time system in an organized way. Start with the fact that a day is divided into twenty-four hours, a day representing a full rotation of the earth on its axis. A clock's face is divided into twelve hours, which means that the hour hand circles the clock twice each day, once for the morning hours, once for the afternoon and evening hours. Explain that a.m. is the abbreviation of the Latin word *ante-meridiem,* which means before noon, and that p.m. is the abbreviation of the Latin word *post-meridiem,* meaning after noon. Noon, or the *meridiem,* is the time of day when the sun is highest in the sky. Before noon the sun rises; after noon it starts going down. Before clocks were invented sundials were used to tell the time of day. The sundial is an instrument that indicates time by the position of the shadow of a pointer cast by the sun on the face of a dial marked in hours. At noon, when the sun is directly above, no shadow is cast to the right or left of the pointer.

The clock has two hands—an hour hand (the short hand) and a minute hand (the longer hand). The hour hand circles the clock every twelve hours, the minute hand circles it every sixty minutes or hour. Some clocks have a second-hand which sweeps around the clock each minute, that is, every sixty seconds.

To tell the correct time we must note the position of the two hands. When both hands are at the twelve position, it is either 12 noon or 12 midnight. The clock itself does not tell us whether it is a.m. or p.m. This we determine by our own observation.

The hour hand, of course, tells us what hour it is. The minute hand tells us how far into the hour we are. Since an hour is composed of sixty minutes, when the minute hand is at the figure 3, it is a quarter past the hour; at the figure 6, it is half past the hour; at position 9, it is three-quarters past the hour, or one quarter before the next hour. In terms of minutes, each position on the clock represents 5 minutes. Thus, the figure 3, or one-quarter past the hour, represents 15 minutes. The figure 6 represents 30 minutes past the hour, and the figure 9 represents 15 minutes before the next hour.

Demonstrate clock positions to the pupil by drawing a clock-face on the blackboard and drawing the hour and minute hands in a variety of positions.

Give the pupil practice problems in telling time, and in converting hours into minutes, minutes into seconds, minutes into fractions of hours, and seconds into fractions of minutes. Devise problems in which time must be calculated. For example, if we leave New York at 11 a.m. and arrive in Boston at 3:25 p.m. how long does the trip take?

Explain the four different time zones in the United States: Eastern Standard Time (E.S.T.), Central Standard Time (C.S.T.), Mountain Standard Time (M.S.T.), and Pacific Standard Time (P.S.T.). Each time zone is one hour earlier going from east to west—because it takes the sun one hour to travel 15° on the earth's surface. Thus, if it is 4 p.m. in New York City, it is 3 p.m. in Chicago, 2 p.m. in Denver, and 1 p.m. in Los Angeles. Thus, if you are flying from New York to Chicago, leaving New York at 3 p.m. E.S.T. and arriving at Chicago at 3:30 p.m. C.S.T., how long does the trip take? The correct answer is one and a half hours, not a half hour.

Daylight-saving time is one hour ahead of standard time, generally used in the summer to give an hour more of daylight at the end of the usual working day. Unless the time is designated as daylight-saving time (D.S.T.) it is standard time. Thus, when we enter daylight-saving time we move the clock forward one hour. When we go off daylight-saving time, we move the clock back an hour. We move forward to daylight-saving, back to standard.

After the pupil has mastered an understanding of our domestic time zones, you can take up the time zones in the rest of the world. This can be done by determining the time in the major cities of the world in relation to the time zone in which the pupil lives. Explain that the world is divided into twenty-four standard time zones calculated east and west of a line drawn through Greenwich, England, a borough of London. This line is known as the Prime Meridian (first meridian). In other words, the twenty-four time zones were determined on the basis of their relationship to noon at Greenwich. Naturally, when it is noon at Greenwich, it is midnight on the other side of the globe. At midnight we pass from one day to the next. On the other side of the globe, the prime meridian becomes the 180th meridian.

For the sake of establishing a universal calendar, the nations of the world agreed to accept a line drawn largely along the 180th meridian in the middle of the Pacific as the International Date Line. Thus, although it may be noon on the 180th meridian it will be 11 a.m. Sunday at the next time zone west of the date line and 1 p.m. Saturday at the next time zone east of the date line. Thus, you lose a day when you travel west across the date line, but you gain a day when you travel east across it. This may seem to be a strange phenomenon to those who never travel out of their time zones. But the earth is constantly rotating, and while most of us remain in the same place, we go from one day to the next in the middle of the same night and think nothing of it. On New Year's Eve, at the stroke of midnight, we go from one year to the next. Nothing visible has occurred, but what has occurred in terms of human understanding is so great an event that celebrations take place to commemorate it. New Year's Eve is the best way to prove that mass hysteria can be induced by a set of arbitrary numbers. The reason for this, of course, is that the passing of time has meaning and that the numbers we use to represent the passage of time take on added significance. Thus, numbers like 1776, 1860, 1914, 1939, 1984 become incredibly powerful stimulators of the mind, in that they conjure up events and cataclysms that have left their mark on human history or will do so in the future.

Step 66: The Calendar. Undoubtedly the pupil will know the days of the week and the months of the year. He will know the four seasons. He will probably know that there are seven days in a week and twelve months in a year. But he may not know how many days or weeks there are in a year.

When we ask how many days there are in a year, we are really asking how many times does the earth rotate on its axis as it completes a full circle or revolution around the sun. The earth rotates 365¼ times each year. But since we cannot have a calendar with a quarter-day, we compute the calendar year as having 365 days, a quarter-day short of a full year. Every four years (leap year) we add an extra day. This extra day is tagged on at the end of February which regularly has 28 days but has 29 in leap years. So every four years we make our arithmetic adjustment in our calendar. If you explain the basis of our calendar in this way, the pupil will understand the meaning of leap year as an arithmetic adjustment. He will also understand the difference between the year as a calendar computation and the actual physical year of the earth's movement around the sun.

To remember the number of days each month has, teach the pupil the well known rhyme: Thirty days hath September,/April, June and November/All the rest have thirty-one/Except February alone,/Which has four and twenty-four/Till leap year brings it one day more.

There are fifty-two weeks and one day in the year, with an extra day in leap years. Ten years make a decade, a hundred years make a century. The years in the 1900's are in the twentieth century. To explain this, show how the years 1 through 100 were the first century, the years 101 through 200 were the second century, etc.

Our present calendar system starts with the birth of Christ, so that the year 1 A.D. means 1 year after the birth of Christ, while 1 B.C. means 1 year before the birth of Christ. Have the pupil compute such problems as the number of years between 3500 B.C. and 1970.

The significance of the calendar is not only that we use it to give our everyday lives a sense of order and continuity, but also to keep track of our history which is crowded with events and people. The calendar reveals a great deal about man's basic nature, his need to fit the order of his life to the order of the physical universe, his need to devise tools with which to aid memory and keep records, his need to live a regulated, nonchaotic existence in which he can plan the future.

The calendar also represents a considerable human achievement in terms of solar-system observance. It was the first practical result of man's astronomical studies in which he could relate the seasonal changes on earth to the changes taking place in the heavens. In those days it was assumed that the earth was stationary but that everything in the heavens moved around it. By recording all these movements and observing their regular recurrence man was able to predict certain future conditions and plan accordingly. The calendar became the primary tool for planning. It is one of the best tools man has ever devised to help him control his environment and plan ahead. It should be noted that all astrological forecasting is based on the calendar in relation to the movements of heavenly bodies. While man has devised more scientific and accurate ways of forecasting the future, astrology remains a popular way to predict the unpredictable.

Teach the child to read dates and to make his own calendar. Everyone's birthday is an important calendar date. Have the pupil relate his own birthday to important world events. Also devise problems using dates; for example, computing the 250th anniversary of the signing of the Declaration of Independence or the age of George Washington if he were alive today. Teach him to make a chronological chart of events, to chart his own program of activities for the coming year. This will show the child how to use the calendar in planning ahead.

In teaching the four seasons, explain the phenomenon of the equinox, that is, the two days each year in which both day and night are of equal length. They occur on the first day of autumn and the first day of spring when the sun's rays are vertical to the earth. The first is called the autumnal equinox and occurs about September 23, the second is called the vernal equinox and occurs about March 21. You can use this lesson on the calendar as a means of stimulating the pupil's interest in the solar system and how mathematics is used to understand the relationship of one heavenly body to another.

Step 67: Intermediary Arithmetic. We have covered all of the arithmetic skills a child is expected to learn in the primary and elementary grades. Some children, obviously, will learn faster than others. The slower children will require explanations in the simplest, most elementary form and will need to have these explanations frequently repeated, with variations in the approach and in the nature of the illustrative examples. The brighter pupils will see reasons, grasp concepts, and understand processes more quickly. But once the child has firmly mastered the material in this course of instruction he can move on to arithmetic on the intermediary levels. What follows is an outline on how to expand the skills already learned.

Addition: The pupil should learn to add longer columns of numbers with a variety of digits, making sure that the right digit is added in the right column. He should perfect his skill in carrying, regardless of how many digits there are in the numbers added.

Subtraction: Here the child should develop his equal adding or borrowing skill in subtractions with numbers of three, four, five, six or more digits.

Multiplication: The pupil should be taught to multiply with both larger multiplicands and larger multipliers. The first step is to teach him to multiply two-digit multiplicands with two-digit multipliers, as in the following example, without carrying:

a.
$$\begin{array}{r} 43 \\ \underline{22} \end{array}$$

b. Multiply the multiplicand with the "ones" digit of the multiplier.

$$\begin{array}{r} 43 \\ \underline{22} \\ 86 \end{array}$$

c. Multiply the multiplicand with the "tens" digit of the multiplier, placing the first written digit—the "ones" digit—of the partial product directly under the "tens" digit of the multiplier. The reason why we must indent the partial product so that it falls directly under the "tens" digit in the multiplier is because we are actually multiplying the multiplicand by 20 not 2, since the 2 in the "tens" column represents multiples of ten in our place-value system. However, we accomplish the same thing more simply by multiplying the multiplicand by 2, not 20, and indenting the partial product so that it falls directly under the "tens" digit of the multiplier.

$$\begin{array}{r} 43 \\ \underline{22} \\ 86 \\ \underline{86} \end{array}$$

d. Add the two partial products for the final product.

$$\begin{array}{r} 4\ 3 \\ \underline{2\ 2} \\ 8\ 6 \\ \underline{8\ 6} \\ 9\ 4\ 6 \end{array}$$

The next example requires carrying.

a. $\begin{array}{r} 69 \\ \underline{23} \end{array}$

b. Multiply the multiplicand with the "ones" digit of the multiplier.

$$\begin{array}{r} 2 \\ 6\ 9 \\ \underline{2\ 3} \\ 2\ 0\ 7 \end{array}$$

c. Multiply the multiplicand with the "tens" digit of the multiplier, indenting the partial product as shown in the previous example. Write the carried digit above the previously carried digit distinctly enough to avoid confusing them.

```
   1
   2
   6 9
   2 3
 -----
 2 0 7
1 3 8
-----
```

d. Add the two partial products to get the final product.

```
    6 9
    2 3
   ----
  2 0 7
1 3 8
-------
1 5 8 7
```

In the next step we multiply three-digit multiplicands with two-digit multipliers, first without carrying, then with carrying:

```
                    2 3
                    3 4
  1 2 3             2 6 8
    2 1               4 5
  -----           -------
  1 2 3           1 3 4 0
2 4 6           1 0 7 2
-------         ---------
2 5 8 3         1 2 0 6 0
```

In the next step we multiply three-digit multiplicands with three-digit multipliers, first without carrying, to demonstrate the process more simply, then with carrying. Note the indention of the "tens" and "hundreds" partial products consistent with place-value notation. You can demonstrate the validity of this method by multiplying 231 by 3, 20, and 100 and adding the three products together.

```
    2 3 1
    1 2 3
    -----
    6 9 3
  4 6 2
2 3 1
---------
2 8 4 1 3
```

When carrying is required in multiplying with each digit of the multiplier, be sure to have the pupil write the "carries" clearly and separately in their own lines to avoid confusion.

```
      1
      4 1
      4 1
      4 5 2
      3 8 9
    -------
      1
    4 0 6 8
  1
  3 6 1 6
1 3 5 6
-----------
1 7 5 8 2 8
```

In the next steps, we multiply four-, five-, and six-digit multiplicands by three-, four-, and five-digit multipliers

Multipliers ending with zeros. When the multiplier is 10, 100, 1,000, etc., the product is obtained by adding the number of zeros in the multiplier to the multiplicand. For example:

4,342 × 10 = 43,420

396 × 1000 = 396,000

42,437 × 100 = 4,243,700

Other examples of multipliers ending with zeros:

a.

```
   345        or     345
   250                250
 17250              17250
 690                690
 86250              86250
```

b.

```
   345        or     345
   200                 200
 69000               69000
```

c.

```
        4 5 6 8       or     4 5 6 8
        3 4 0 0                    3 4 0 0
  1 8 2 7 2 0 0              1 8 2 7 2 0 0
1 3 7 0 4                  1 3 7 0 4
1 5,5 3 1,2 0 0            1 5,5 3 1,2 0 0
```

d.

```
        4 5 6 8       or     4 5 6 8
        6 0 0 0                    6 0 0 0
2 7,4 0 8,0 0 0            2 7,4 0 8,0 0 0
```

Checking multiplication. Multiplication can be checked by interchanging the multiplicand and multiplier and comparing products, which should be identical.

Division: Developing the ability to divide with two or more digit divisors requires developing considerable numerical judgment. This judgment consists of the ability to mentally estimate how many times a divisor can go into a dividend and the successive digits in the quotient. Such numerical judgment can only be developed with a great deal of practice. Such practice will enable a pupil to see, for example, that 22 goes into 90 four times with a remainder of 2. If the divisor is 27 and the dividend 96, the pupil will estimate the quotient by thinking of 30 into 90. His final answer will be a quotient of 3 with a remainder of 15. Here is how these problems appear on paper:

$$\begin{array}{rr} & 4 \\ 22\big) & \overline{90} \\ & \underline{88} \\ & 2 \end{array} \qquad\qquad \begin{array}{rr} & 3 \\ 27\big) & \overline{96} \\ & \underline{81} \\ & 15 \end{array}$$

In developing the ability to divide with larger divisors, first give the pupil exercises with two-digit divisors and two-digit dividends, then two-digit divisors with three-, four-, five-, and six-digit dividends.

Next, advance to three-digit divisors with three-, four-, five-, six-, and seven-digit dividends. From there, provide exercises with four-digit divisors and appropriate dividends. Once the pupil has mastered the skills learned in solving these problems, he should have no trouble dividing any size dividend with any size divisor.

Examples:

Two-digit divisor with a three-digit dividend.

a. $$58\big)\overline{849}$$

b. $$\begin{array}{rl} & \ \ 1 \\ 58\big) & \overline{849} \\ & \underline{58} \\ & 26 \end{array}$$

c. Bring down 9. Estimate 58 into 269 as 50 into 250.

$$\begin{array}{rl} & \ \ 15 \\ 58\big) & \overline{849} \\ & \underline{58} \\ & 269 \\ & 290 \end{array}$$

d. 5 is too large. Try 4.

```
     14
58)849
   58
   269
   232
    37
```

e. Final answer is quotient of 14 with remainder of 37

Note how numerical judgment only provides us with an estimate. The final correct answer is arrived at by trying different numbers. It is like trying on different sizes of shoes to see which one will fit.

Provide sufficient exercises to help the pupil develop numerical judgment.

Two-digit divisor with a four-digit dividend:

a.

```
39)8754
```

b.

```
    2
39)8754
   78
    9
```

c. Bring down 5. 39 goes into 95 twice.

```
    22
39)8754
   78
    95
    78
    17
```

d. Bring down 4. 39 goes into 174 five times?

```
     225
39)8754
   78
    95
    78
    174
    195
```

e. Five is too large. Try 4.

```
     224
39)8754
   78
    95
    78
    174
    156
     18
```

Two-digit divisor with a four-digit dividend in which the divisor is larger than the first two digits of the dividend:

a.

```
86)6387
```

b. Since 86 goes into 63 less than one time, we must divide 638 by 86. Our numerical judgment suggests that we try 7.

```
       7
86)6387
   602
    36
```

c. Bring down 7. 86 goes into 367 four times?

```
      74
86)6387
   602
    367
    344
     23
```

d. Correct answer: quotient of 74, remainder 23.

Two-digit divisor with five-digit dividend:

a.

```
92)62430
```

b.

```
      6
92)62430
   552
    72
```

c.

```
     67
92)62430
   552
    723
    644
     79
```

d.

```
     678
92)62430
   552
    723
    644
     790
     736
      54
```

Two-digit divisor with a six-digit dividend:

a.

```
48)328,407
```

b.

```
     6
48)328,407
   288
    40
```

c.

```
     6,8
48)328,407
   288
    404
    384
     20
```

d.

```
     6,84
48)328,407
   288
    404
    384
     200
     192
       8
```

e.

```
     6,841
48)328,407
   288
    404
    384
     200
     192
       87
       48
       39
```

Three-digit divisor with a four-digit dividend:

a.

```
609)8661
```

b.

```
       1
609)8661
    609
    257
```

c.

```
      14
609)8661
    609
    2571
    2436
     135
```

Three-digit divisor with a five-digit dividend:

a.

```
589)25890
```

b.

```
        4
589)25890
    2356
     233
```

c.

```
       43
589)25890
    2356
     2330
     1767
      563
```

Three-digit divisor with a six-digit dividend:

a.

```
357)892,600
```

b.

```
       2
357)892,600
    714
    178
```

c.

```
       2,5
357)892,600
    714
    1786
    1785
       1
```

d.

```
       2,500
357)892,600
    714
    1786
    1785
       100
```

Three-digit divisor with a seven-digit dividend:

a.

```
920)5,780,325
```

b.

```
         6,
920)5,780,325
    5 520
      260
```

c.

```
          6,2
920)5,780,325
    5 520
      2603
      1840
       763
```

d.

```
          6,28
920)5,780,325
    5 520
      2603
      1840
       7632
       7360
        272
```

e.

```
          6,283
920)5,780,325
    5 520
      2603
      1840
       7632
       7360
        2725
        2760
```

f. Three is too large. Try 2.

```
          6,282
920)5,780,325
    5 520
      2603
      1840
       7632
       7360
        2725
        1840
         885
```

Notice all the skills required to perform long division: division, multiplication, higher decade additions with the multiplication "carries," and subtraction with equal adding or "borrowing." Long division, in fact gives the pupil lots of practice in most of the arithmetic skills he has learned. However, it is essential to see where the pupil is strongest and weakest in his use of these skills. In this way his weak points can be strengthened by remedial drills and exercises.

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